- 作业讲解
  - JH第4章练习4.3.6.6

JH第4章练习4.3.6.6

- 0-1背包问题的拉格朗日近似（与本题无关）

$$\text{maximize} \quad z = \sum_{j=1}^{n} p_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} w_j x_j \le c,$$

$$x_j = 0 \text{ or } 1, \qquad j \in N = \{1, \ldots, n\},$$

where

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected;} \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow$$

$$\text{maximize} \quad \sum_{j=1}^{n} p_j x_j + \lambda \left(c - \sum_{j=1}^{n} w_j x_j \right)$$

$$\text{subject to} \quad x_j = 0 \text{ or } 1, \qquad j = 1, \ldots, n.$$

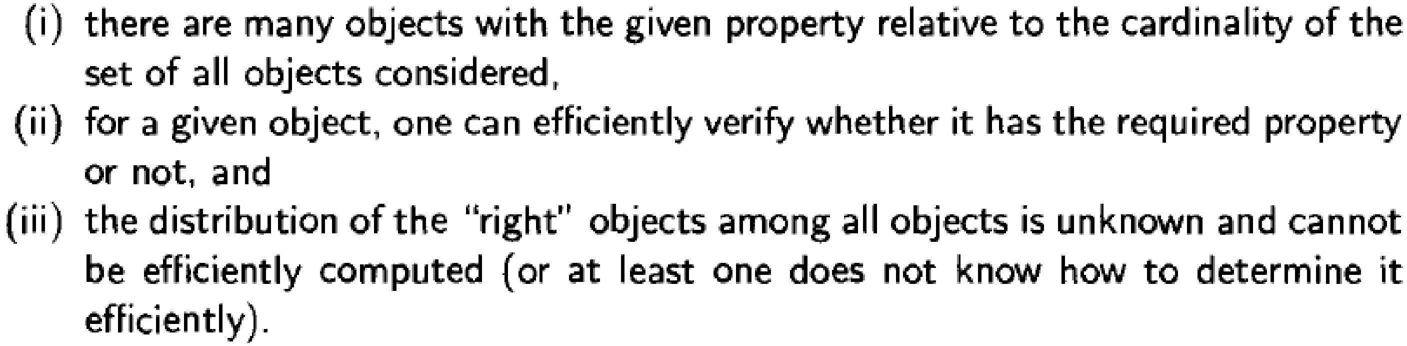
当λ≥0时，新问题的解是原问题的解的上界。
→ 选取不同的λ，新问题有不同的最优解，最小的那个即原问题的最优解。
→ 问题转变为：如何找到让新问题目标函数值最小的那个λ?
这本身是一个最小化问题，并且优化目标是一个分段线性的凸函数，解法略。

- 教材讨论
  - JH第5章第3节第1、2、3小节

问题1：random sampling and Las Vegas

- 什么是random sampling？
- 什么样的问题适合采用random sampling？
  - 因此，为什么quadratic residues适合采用random sampling？

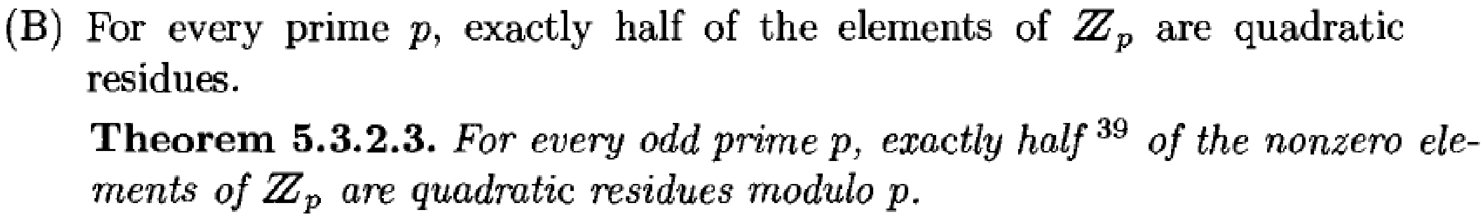
问题1：random sampling and Las Vegas (续)



(i) there are many objects with the given property relative to the cardinality of the set of all objects considered,

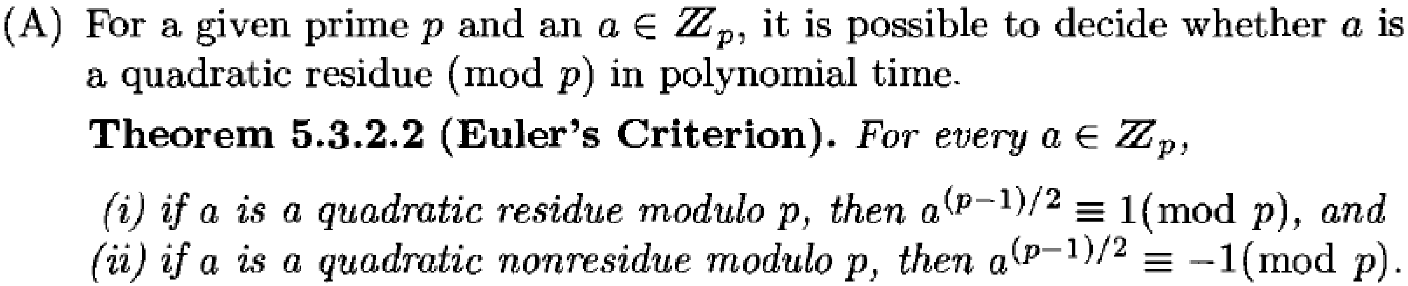
(ii) for a given object, one can efficiently verify whether it has the required property or not, and

(iii) the distribution of the "right" objects among all objects is unknown and cannot be efficiently computed (or at least one does not know how to determine it efficiently).



(B) For every prime p, exactly half of the elements of $\mathbb{Z}_p$ are quadratic residues.

Theorem 5.3.2.3. *For every odd prime p, exactly half*[39] *of the nonzero elements of $\mathbb{Z}_p$ are quadratic residues modulo p.*

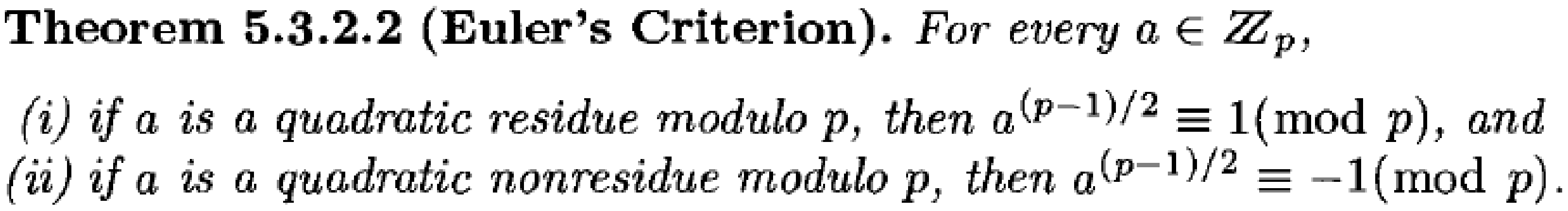


(A) For a given prime p and an $a \in \mathbb{Z}_p$, it is possible to decide whether a is a quadratic residue (mod p) in polynomial time.

Theorem 5.3.2.2 (Euler's Criterion). *For every $a \in \mathbb{Z}_p$,*

(i) if a is a quadratic residue modulo p, then $a^{(p-1)/2} \equiv 1 (\text{mod } p)$, and
(ii) if a is a quadratic nonresidue modulo p, then $a^{(p-1)/2} \equiv -1 (\text{mod } p)$.

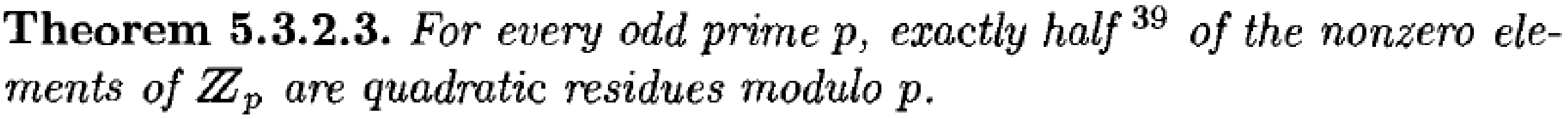
问题1：random sampling and Las Vegas (续)



Theorem 5.3.2.2 (Euler's Criterion). *For every* $a \in \mathbb{Z}_p$,

(i) if a *is a quadratic residue modulo* p, *then* $a^{(p-1)/2} \equiv 1 (\mathrm{mod}\ p)$, *and*
(ii) if a *is a quadratic nonresidue modulo* p, *then* $a^{(p-1)/2} \equiv -1 (\mathrm{mod}\ p)$.

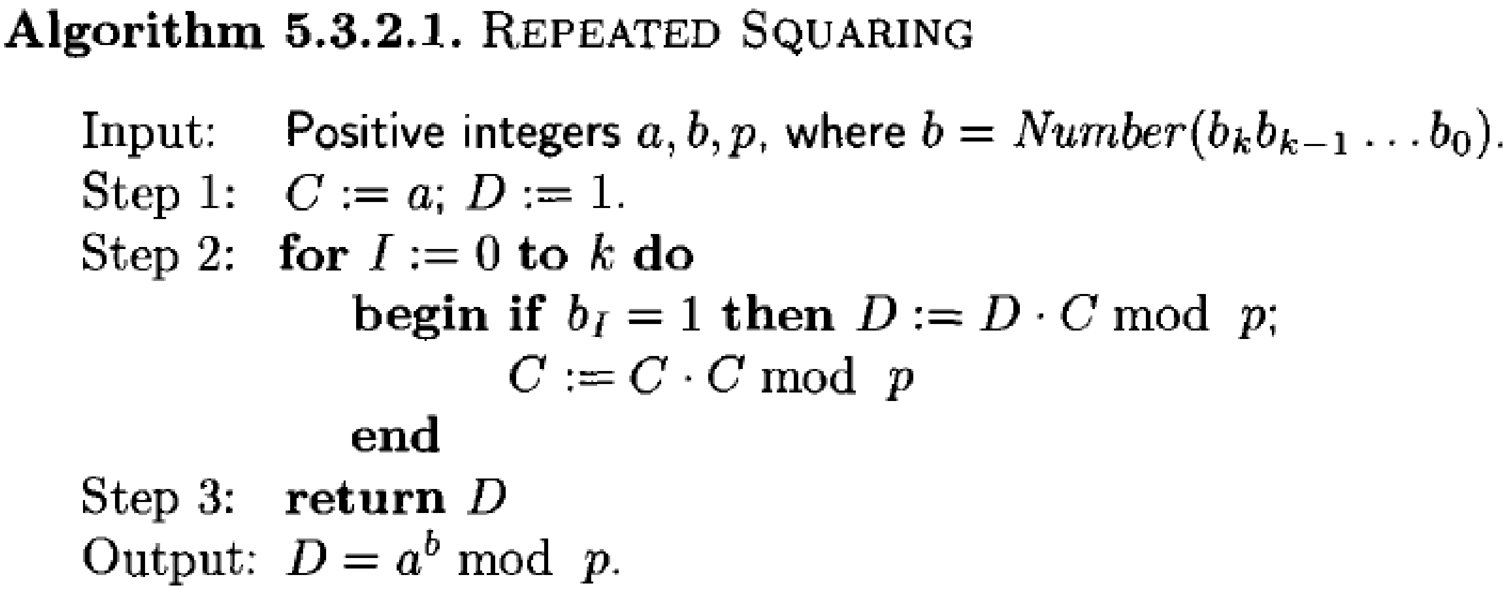
- 判定quadratic residue的时间复杂度是多少？为什么？



Theorem 5.3.2.3. *For every odd prime* p, *exactly half*[39] *of the nonzero elements of* $\mathbb{Z}_p$ *are quadratic residues modulo* p.

- 你能解释这个定理的证明思路吗？（主要分为哪两步）

问题1： random sampling and Las Vegas (续)

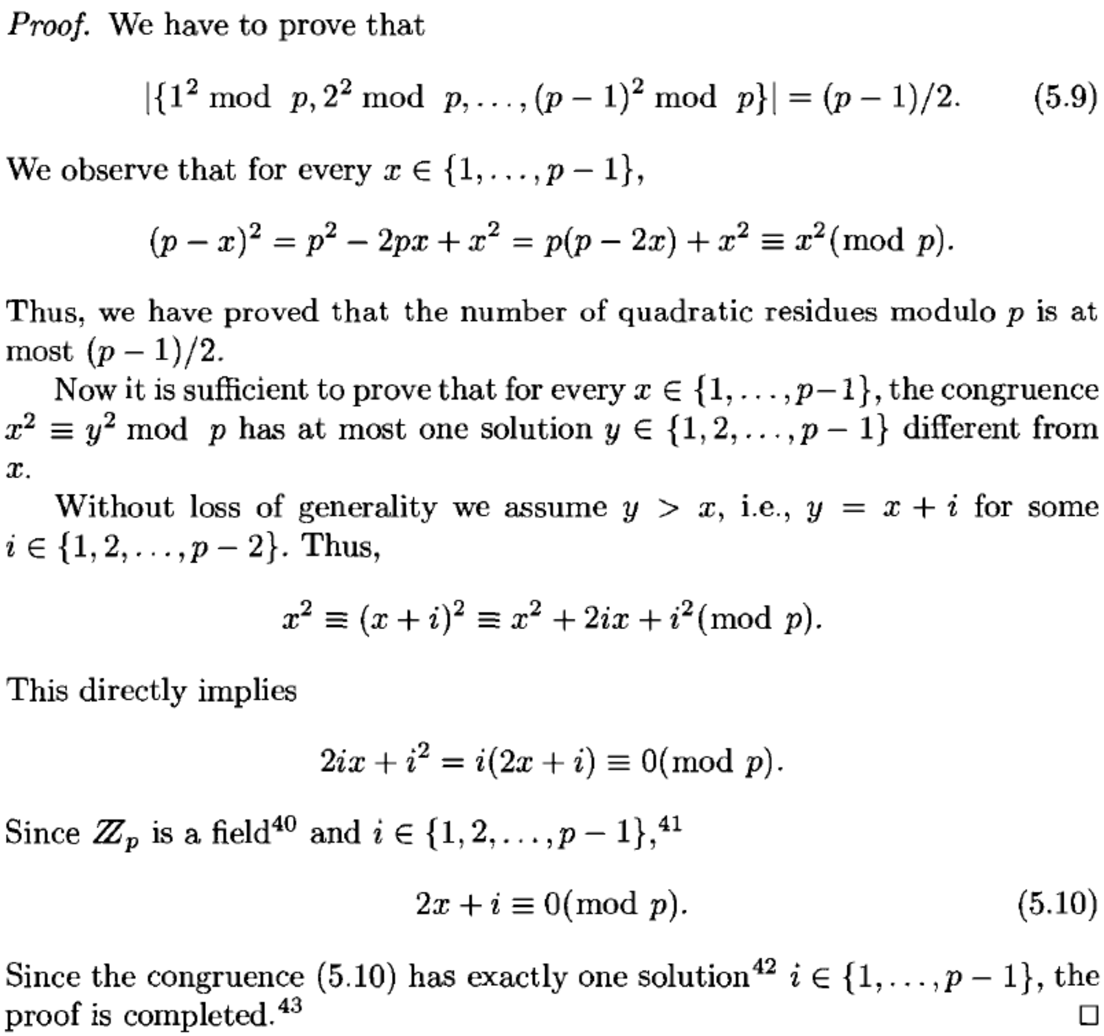


Algorithm 5.3.2.1. REPEATED SQUARING

```
Input:   Positive integers $a, b, p$, where $b = Number(b_k b_{k-1} \ldots b_0)$.
Step 1:  $C := a$; $D := 1$.
Step 2:  for $I := 0$ to $k$ do
             begin if $b_I = 1$ then $D := D \cdot C \bmod p$;
                   $C := C \cdot C \bmod p$
             end
Step 3:  return $D$
Output:  $D = a^b \bmod p$.
```

$$\boldsymbol{O\left(2(k+1)\times k^2\right)}$$

问题1：random sampling and Las Vegas (续)



Proof. We have to prove that

$$|\{1^2 \bmod p, 2^2 \bmod p, \ldots, (p-1)^2 \bmod p\}| = (p-1)/2. \qquad (5.9)$$

We observe that for every $x \in \{1, \ldots, p-1\}$,

$$(p-x)^2 = p^2 - 2px + x^2 = p(p-2x) + x^2 \equiv x^2 (\text{mod } p).$$

① Thus, we have proved that the number of quadratic residues modulo p is at most $(p-1)/2$.

② Now it is sufficient to prove that for every $x \in \{1, \ldots, p-1\}$, the congruence $x^2 \equiv y^2 \bmod p$ has at most one solution $y \in \{1, 2, \ldots, p-1\}$ different from x.

Without loss of generality we assume $y > x$, i.e., $y = x + i$ for some $i \in \{1, 2, \ldots, p-2\}$. Thus,

$$x^2 \equiv (x+i)^2 \equiv x^2 + 2ix + i^2 (\text{mod } p).$$

This directly implies

$$2ix + i^2 = i(2x+i) \equiv 0 (\text{mod } p).$$

Since $\mathbb{Z}_p$ is a field[40] and $i \in \{1, 2, \ldots, p-1\}$,[41]

$$2x + i \equiv 0 (\text{mod } p). \qquad (5.10)$$

Since the congruence (5.10) has exactly one solution[42] $i \in \{1, \ldots, p-1\}$, the proof is completed.[43] □

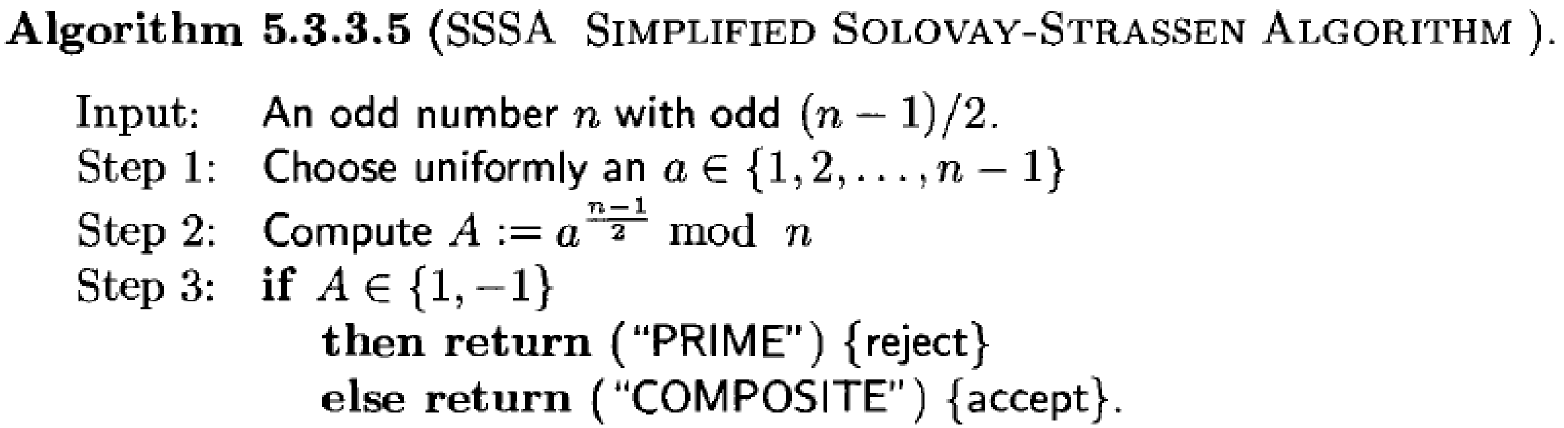
问题2： abundance of witnesses and one-sided-error Monte Carlo

- 作为一个单边错Monte Carlo算法，SSSA是识别质数的还是识别合数的？这两种说法有区别吗？Solovay-Strassen呢？

- 为什么SSSA是一个单边错Monte Carlo算法？
- 定理5.3.3.1的基本证明思路是什么？
- SSSA在使用时的局限是什么？为什么这一局限难以打破？

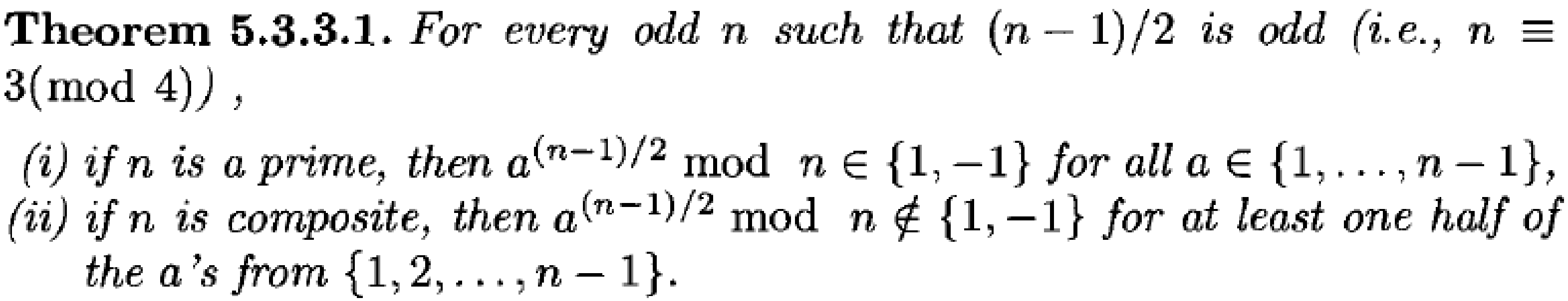
问题2： abundance of witnesses and one-sided-error Monte Carlo (续)

- 为什么SSSA是一个单边错Monte Carlo算法？



Algorithm 5.3.3.5 (SSSA SIMPLIFIED SOLOVAY-STRASSEN ALGORITHM).

```
Input:   An odd number $n$ with odd $(n-1)/2$.
Step 1:  Choose uniformly an $a \in \{1, 2, \ldots, n-1\}$
Step 2:  Compute $A := a^{\frac{n-1}{2}} \bmod n$
Step 3:  if $A \in \{1, -1\}$
             then return ("PRIME") {reject}
             else return ("COMPOSITE") {accept}.
```

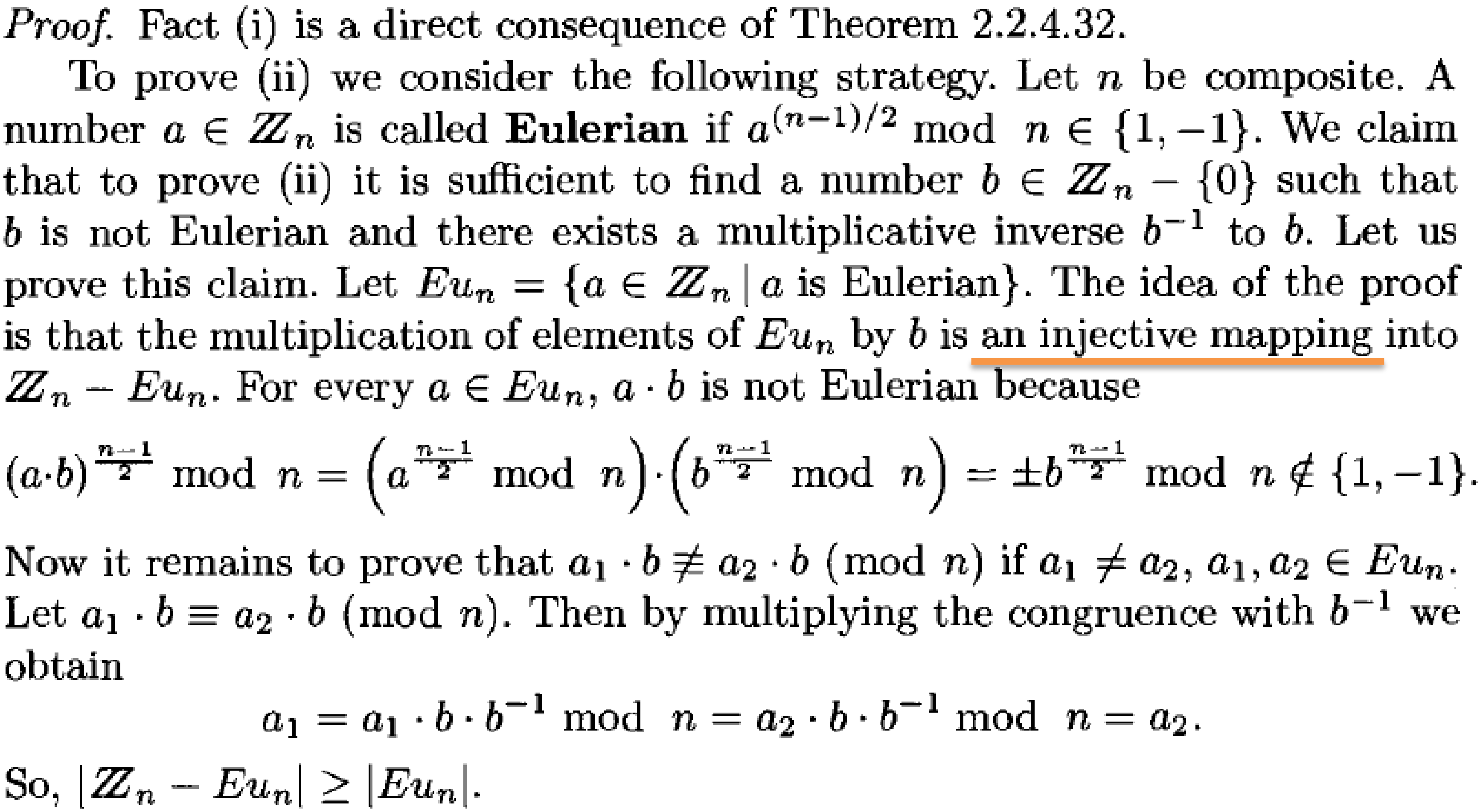


Theorem 5.3.3.1. *For every odd n such that $(n-1)/2$ is odd (i.e., $n \equiv 3 \pmod 4$)* ,

(i) if n is a prime, then $a^{(n-1)/2} \bmod n \in \{1, -1\}$ for all $a \in \{1, \ldots, n-1\}$,

(ii) if n is composite, then $a^{(n-1)/2} \bmod n \notin \{1, -1\}$ for at least one half of the a's from $\{1, 2, \ldots, n-1\}$.

问题2： abundance of witnesses and one-sided-error Monte Carlo (续)



Proof. Fact (i) is a direct consequence of Theorem 2.2.4.32.

To prove (ii) we consider the following strategy. Let n be composite. A number $a \in \mathbb{Z}_n$ is called **Eulerian** if $a^{(n-1)/2} \bmod n \in \{1, -1\}$. We claim that to prove (ii) it is sufficient to find a number $b \in \mathbb{Z}_n - \{0\}$ such that b is not Eulerian and there exists a multiplicative inverse b^{-1} to b. Let us prove this claim. Let $Eu_n = \{a \in \mathbb{Z}_n \mid a \text{ is Eulerian}\}$. The idea of the proof is that the multiplication of elements of Eu_n by b is an injective mapping into $\mathbb{Z}_n - Eu_n$. For every $a \in Eu_n$, $a \cdot b$ is not Eulerian because

$$(a \cdot b)^{\frac{n-1}{2}} \bmod n = \left(a^{\frac{n-1}{2}} \bmod n\right) \cdot \left(b^{\frac{n-1}{2}} \bmod n\right) = \pm b^{\frac{n-1}{2}} \bmod n \notin \{1, -1\}.$$

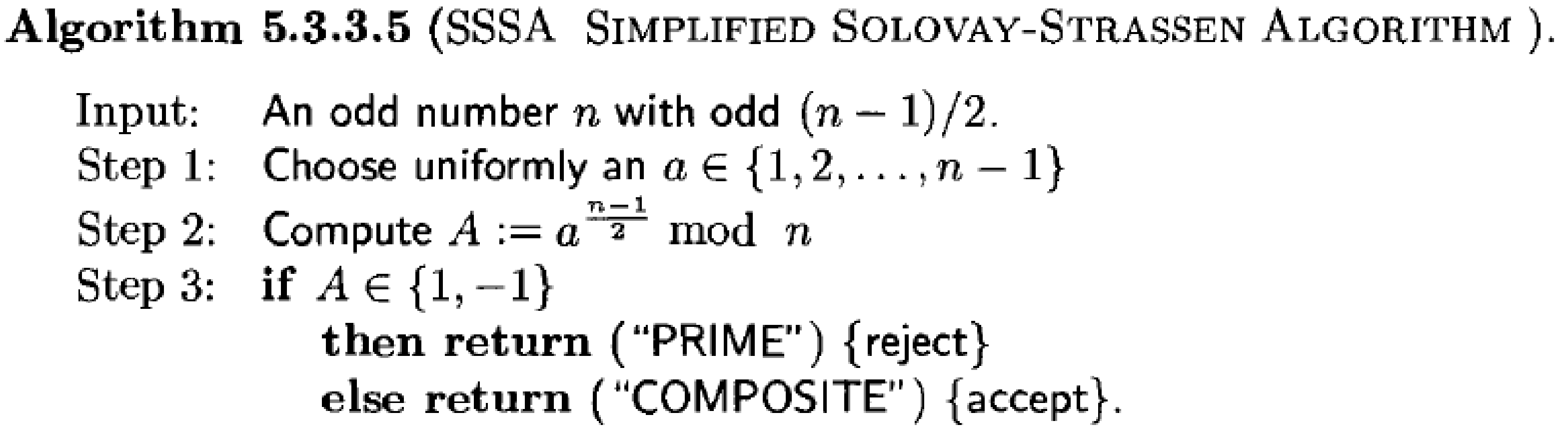
Now it remains to prove that $a_1 \cdot b \not\equiv a_2 \cdot b \pmod{n}$ if $a_1 \neq a_2$, $a_1, a_2 \in Eu_n$. Let $a_1 \cdot b \equiv a_2 \cdot b \pmod{n}$. Then by multiplying the congruence with b^{-1} we obtain

$$a_1 = a_1 \cdot b \cdot b^{-1} \bmod n = a_2 \cdot b \cdot b^{-1} \bmod n = a_2.$$

So, $|\mathbb{Z}_n - Eu_n| \geq |Eu_n|$.

问题2： abundance of witnesses and one-sided-error Monte Carlo (续)

- SSSA在使用时的局限是什么？为什么这一局限难以打破？



Algorithm 5.3.3.5 (SSSA SIMPLIFIED SOLOVAY-STRASSEN ALGORITHM).

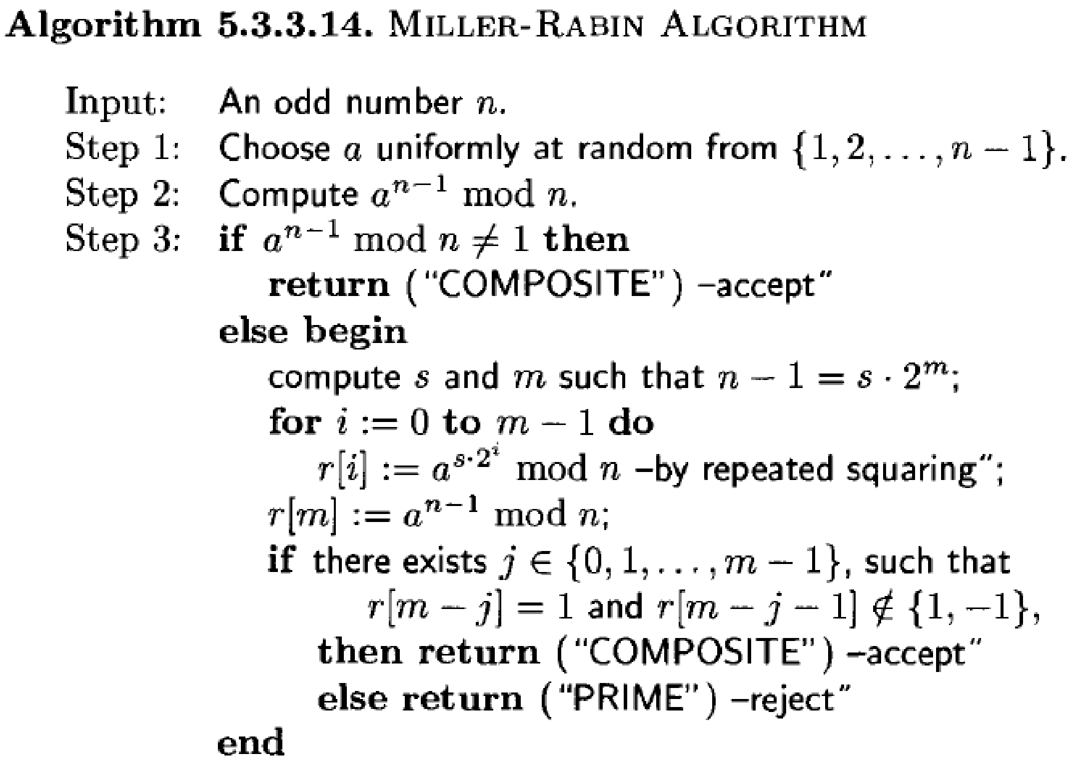
```
Input:   An odd number $n$ with odd $(n-1)/2$.
Step 1:  Choose uniformly an $a \in \{1, 2, \ldots, n-1\}$
Step 2:  Compute $A := a^{\frac{n-1}{2}} \bmod n$
Step 3:  if $A \in \{1, -1\}$
             then return ("PRIME") {reject}
             else return ("COMPOSITE") {accept}.
```

Carmichael numbers:

$$a^{n-1} \equiv 1 (\bmod\ n) \text{ for all } a \in \{1, 2, \ldots, n-1\} \text{ with } gcd(a, n) = 1.$$

问题2： abundance of witnesses and one-sided-error Monte Carlo (续)

- Miller-Rabin的基本原理是什么？

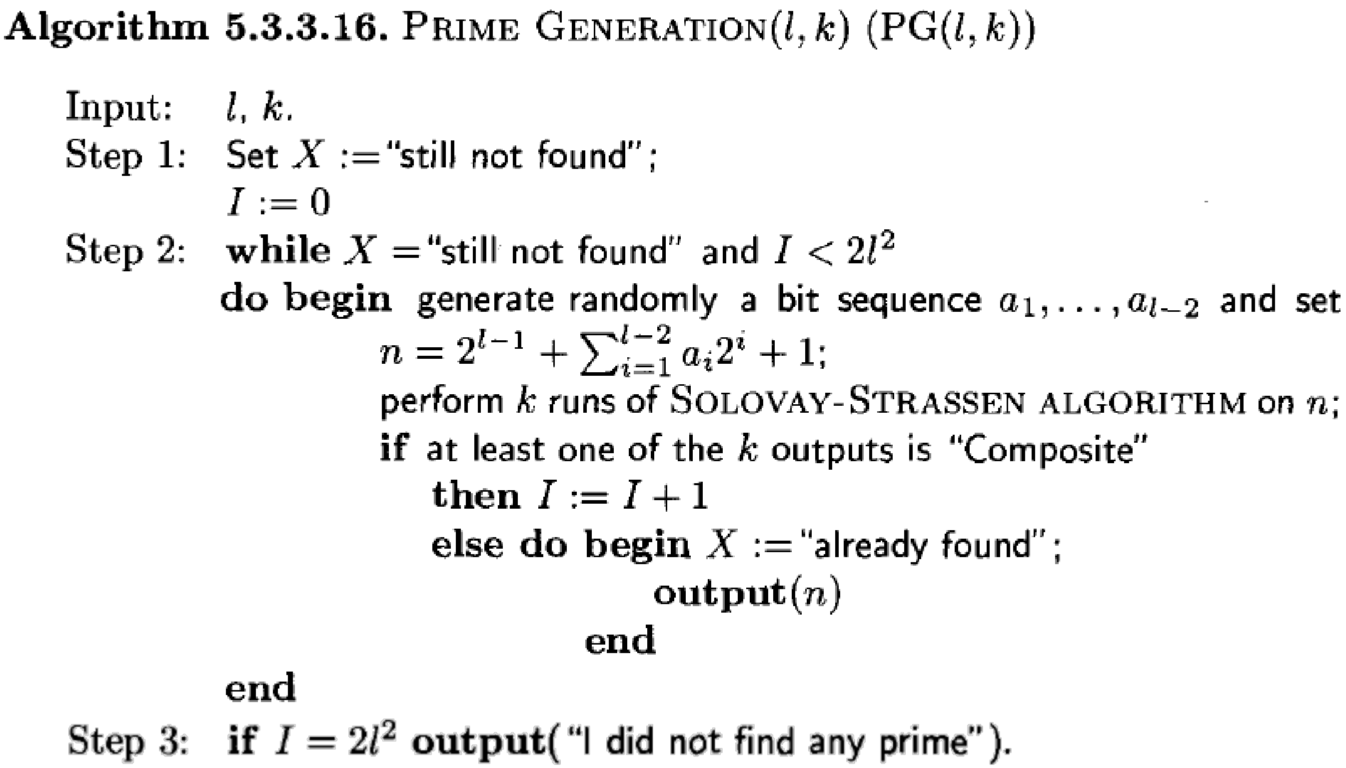


Algorithm 5.3.3.14. MILLER-RABIN ALGORITHM

```
Input:   An odd number $n$.
Step 1:  Choose $a$ uniformly at random from $\{1, 2, \ldots, n-1\}$.
Step 2:  Compute $a^{n-1} \bmod n$.
Step 3:  if $a^{n-1} \bmod n \neq 1$ then
             return (“COMPOSITE”) –accept”
         else begin
             compute $s$ and $m$ such that $n - 1 = s \cdot 2^m$;
             for $i := 0$ to $m - 1$ do
                 $r[i] := a^{s \cdot 2^i} \bmod n$ –by repeated squaring”;
             $r[m] := a^{n-1} \bmod n$;
             if there exists $j \in \{0, 1, \ldots, m-1\}$, such that
                     $r[m-j] = 1$ and $r[m-j-1] \notin \{1, -1\}$,
                 then return (“COMPOSITE”) –accept”
                 else return (“PRIME”) –reject”
         end
```

问题2： abundance of witnesses and one-sided-error Monte Carlo (续)

- 算法5.3.3.16的基本原理是什么？
- 为什么它几乎总能输出正确的结果？证明过程中两个概率算式的含义分别是什么？



Algorithm 5.3.3.16. PRIME GENERATION(l, k) (PG(l, k))

```
Input:   $l$, $k$.
Step 1:  Set $X$ := "still not found";
         $I := 0$
Step 2:  while $X$ = "still not found" and $I < 2l^2$
         do begin generate randomly a bit sequence $a_1, \ldots, a_{l-2}$ and set
                 $n = 2^{l-1} + \sum_{i=1}^{l-2} a_i 2^i + 1$;
                 perform $k$ runs of SOLOVAY-STRASSEN ALGORITHM on $n$;
                 if at least one of the $k$ outputs is "Composite"
                    then $I := I + 1$
                    else do begin $X$ := "already found";
                                  output($n$)
                         end
         end
Step 3:  if $I = 2l^2$ output("I did not find any prime").
```

问题2： abundance of witnesses and one-sided-error Monte Carlo (续)

- Probability of outputting “I did not find any prime”

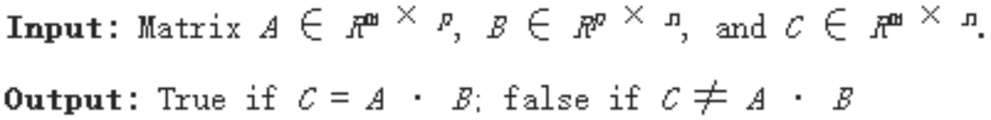
$$\left[\left(1-\frac{1}{2l}\right)\cdot\left(1-\frac{1}{2^k}\right)\right]^{2l^2} < \left(1-\frac{1}{2l}\right)^{2l^2} = \left[\left(1-\frac{1}{2l}\right)^{2l}\right]^{l} < \left(\frac{1}{e}\right)^{l} = e^{-l}.$$

- Probability of outputting a composite number

$$\left(1-\frac{1}{2l}\right)\cdot\frac{1}{2^l}+\sum_{i=1}^{2l^2-1}\left[\left(1-\frac{1}{2l}\right)\cdot\left(1-\frac{1}{2^l}\right)\right]^i\cdot\left(1-\frac{1}{2l}\right)\cdot\frac{1}{2^l}$$

$$\begin{aligned}&\le\left(1-\frac{1}{2l}\right)\cdot\frac{1}{2^l}\cdot\left(\sum_{i=1}^{2l^2-1}\left(1-\frac{1}{2l}\right)^i+1\right)\\&\le\left(1-\frac{1}{2l}\right)\cdot\frac{1}{2^l}\cdot 2l^2\le\frac{l^2}{2^{l-1}}.\end{aligned}$$

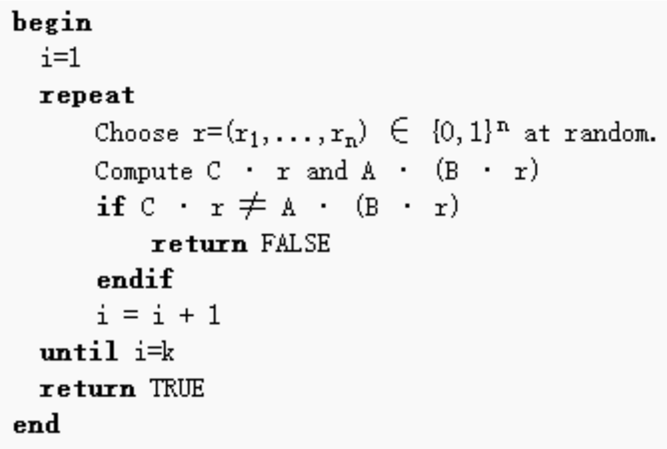
问题2： abundance of witnesses and one-sided-error Monte Carlo (续)



Input: Matrix $A \in R^{m \times p}$, $B \in R^{p \times n}$, and $C \in R^{m \times n}$.

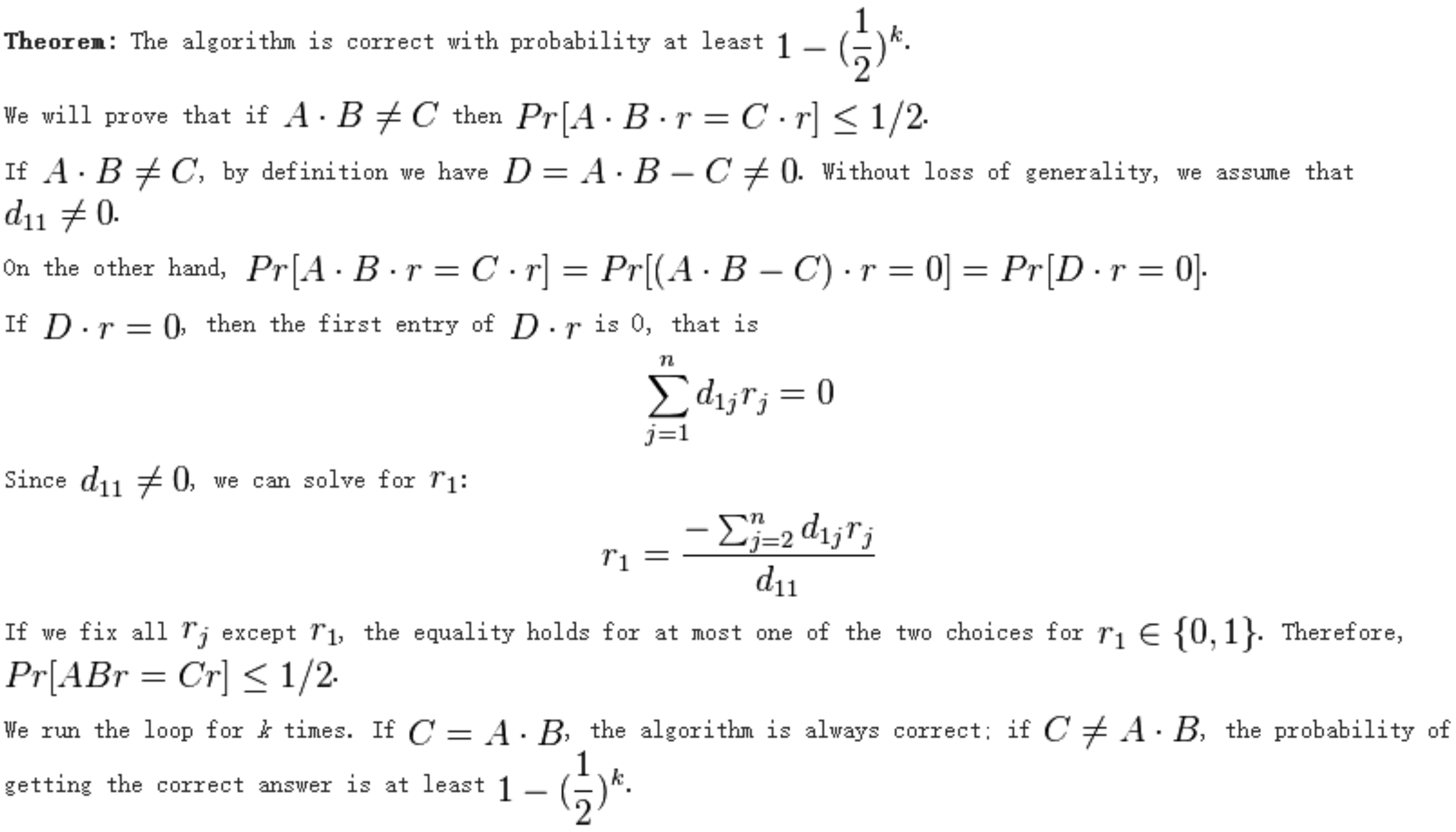
Output: True if $C = A \cdot B$; false if $C \neq A \cdot B$

- 你能不能基于abundance of witnesses给出一个单边错Monte Carlo算法？



```
begin
  i=1
  repeat
      Choose $r=(r_1,...,r_n) \in \{0,1\}^n$ at random.
      Compute $C \cdot r$ and $A \cdot (B \cdot r)$
      if $C \cdot r \neq A \cdot (B \cdot r)$
          return FALSE
      endif
      i = i + 1
  until i=k
  return TRUE
end
```

问题2：abundance of witnesses and one-sided-error Monte Carlo (续)



Theorem: The algorithm is correct with probability at least $1-(\frac{1}{2})^k$.

We will prove that if $A \cdot B \neq C$ then $Pr[A \cdot B \cdot r = C \cdot r] \leq 1/2$.

If $A \cdot B \neq C$, by definition we have $D = A \cdot B - C \neq 0$. Without loss of generality, we assume that $d_{11} \neq 0$.

On the other hand, $Pr[A \cdot B \cdot r = C \cdot r] = Pr[(A \cdot B - C) \cdot r = 0] = Pr[D \cdot r = 0]$.

If $D \cdot r = 0$, then the first entry of $D \cdot r$ is 0, that is

$$\sum_{j=1}^{n} d_{1j} r_j = 0$$

Since $d_{11} \neq 0$, we can solve for r_1:

$$r_1 = \frac{-\sum_{j=2}^{n} d_{1j} r_j}{d_{11}}$$

If we fix all r_j except r_1, the equality holds for at most one of the two choices for $r_1 \in \{0,1\}$. Therefore, $Pr[ABr = Cr] \leq 1/2$.

We run the loop for k times. If $C = A \cdot B$, the algorithm is always correct; if $C \neq A \cdot B$, the probability of getting the correct answer is at least $1-(\frac{1}{2})^k$.