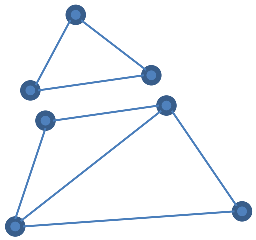
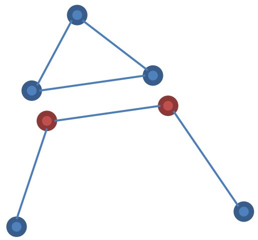
- 作业讲解
  - CS第4.1节问题16、17
  - CS第4.2节问题8、11、17
  - CS第4.3节问题9、13、16
  - CS第4.4节问题1、4、6
  - CS第4.5节问题8、9、10

CS第4.1节问题16

- … This version does not specify that the ears are nonadjacent. What happens if we try proving this by induction, using the same decomposition that we used in proving the Ear Lemma?





CS第4.1节问题17

- … relationship between the number of vertices in a polygon and the number of triangles in any triangulation of that polygon …
  - 在数学归纳法中，使用top-down而非bottom-up的表述方式，即从要证明的结论开始
  - 例如：将任意n边形拆分，而不是从任意n-1边形拼接

CS第4.2节问题8

- $T(n)=2T(n-1)+2000 \quad (n>1)$
- $T(1)=2000$

CS第4.2节问题11

- 定理4.5

CS第4.2节问题17

- 定理4.5
- 定理4.6

$$T(n) = r^n + \sum_{r=1}^{n} r^{n-i} i = r^n + r^n \sum_{r=1}^{n} i \left(\frac{1}{r} \right)^i = \ldots$$

CS第4.3节问题9

- (c)中O($n^{\log_2 3}$)≠O(n^{lg3})

CS第4.3节问题16

- $\Theta(a^n) \neq \Theta(c^n)$
- $\Theta(a^n) \neq \Theta((ac)^n)$

CS第4.4节问题1

- 定理4.11（主定理的扩展形式）

CS第4.5节问题8

- 错误1
  - 欲证$T(n) \le cn^3$
  - 归纳假设$T(n/2) \le c(n/2)^3$
  - 计算$T(n) \le cn^3 + n\lg n = O(n^3) + O(n^2) = O(n^3)$，得证
- 错误2
  - 欲证$T(n) \le cn^3 - dn\lg n$
  - 归纳假设$T(n/2) \le c(n/2)^3 - d(n/2)\lg(n/2)$
  - 计算$T(n) \le \ldots \le cn^3$，得证
- 错误3
  - 欲证$T(n) \le c_1 n^3$
  - 归纳假设$T(n/2) \le c_1 (n/2)^3$
  - 计算$T(n) \le c_1 n^3 + n\lg n \le c_1 n^3 + c_2 n^3 = c_3 n^3$，得证
- 错误4
  - 欲证$T(n) \le cn^3 - d$
  - 归纳假设$T(n/2) \le c(n/2)^3 - d$
  - 计算$T(n) \le cn^3 - d - 7d + n\lg n$，只需取$d \ge n\lg n / 7$，得证

CS第4.5节问题8 (续)

- 一种证法
  - 欲证$T(n) \le c(n^3-n^2)+d$
  - 归纳假设$T(n/2) \le c((n/2)^3-(n/2)^2)+d$
  - 计算$T(n) \le cn^3-2cn^2+8d+n\lg n=[c(n^3-n^2)+d]+(n\lg n+7d-cn^2)$

    $\le [c(n^3-n^2)+d]+(n^2+7d-cn^2)$
  - 只要$c \ge 7d+1$，$T(n) \le [c(n^3-n^2)+d]+(n^2+7d-(7d+1)n^2)$

    $=[c(n^3-n^2)+d]+7d(1-n^2) \le c(n^3-n^2)+d$
  - 并且，$T(1)=d \le d$也成立

- 教材讨论
  - TC第5章
  - CS第5章第6、7节

问题1：randomized algorithm

- 什么样的算法可以称作randomized algorithm？

- 什么叫做randomized algorithm的expected running time？
- 它和average-case running time有什么异同？

问题1：randomized algorithm

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  - Its behavior is determined not only by its input but also by something chosen randomly (e.g. values produced by a random-number generator).

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- 它和average-case running time有什么异同？
  - 异：We discuss the average-case running time when the probability distribution is over the inputs to the algorithm, and we discuss the expected running time when the algorithm itself makes random choices.

问题1： randomized algorithm (续)

- 你能想到哪些方法生成一个32-bit的（伪）随机数？

问题1：randomized algorithm (续)

- 你能想到哪些方法生成一个32-bit的（伪）随机数？
  - Computational methods (pseudo-random number generators)

```
m_w = <choose-initializer>;    /* must not be zero */
m_z = <choose-initializer>;    /* must not be zero */

uint get_random()
{
    m_z = 36969 * (m_z & 65535) + (m_z >> 16);
    m_w = 18000 * (m_w & 65535) + (m_w >> 16);
    return (m_z << 16) + m_w;  /* 32-bit result */
}
```

  - Physical methods
    - Coin flipping
    - Dice
    - Variations in the amplitude of atmospheric noise recorded with a normal radio

问题1： randomized algorithm (续)

- 你能想到哪些方法对一个数组中的元素随机排序？
  你如何评价这样一个方法的好坏？

问题1： randomized algorithm (续)

- 你能想到哪些方法对一个数组中的元素随机排序？
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  - PERMUTE-BY-SORTING(A)

```
n = A.length
let P[1..n] be a new array
for i = 1 to n
    P[i] = RANDOM(1, n^3)
sort A, using P as sort keys
```

  - RANDOMIZE-IN-PLACE(A)

```
n = A.length
for i = 1 to n
    swap A[i] with A[RANDOM(i, n)]
```

问题2： expected running time

- 目前为止，你掌握了哪些方式计算E(X)？

问题2： expected running time

- 目前为止，你掌握了哪些方式计算E(X)？
  - $E(X)=\sum xP(X=x)$ // 定义
  - $E(X)= \sum E(X_i)$ // indicator random variable
  - $E(aX+bY)=aE(X)+bE(Y)$ // linearity of expectation
  - $E(X)=\sum E(X|F_i)P(F_i)$ // conditional expected value

问题2： expected running time (续)

- 你能解释这里用的是哪种方法吗？

```
Slower Quicksort(A,n)
if (n = 1)
    return the one item in A
else
    Repeat
        p = randomElement(A)
        Let H be the set of elements greater than p; Let h = |H|
        Let L be the set of elements less than or equal to p; Let ℓ = |L|
    Until (|H| ≥ n/4) and (|L| ≥ n/4)
    A1 = QuickSort(H,h)
    A2 = QuickSort(L,ℓ)
    return the concatenation of A1 and A2
```

$$T(n) \le E(r)bn + T(a_n n) + T((1-a_n)n)$$

问题2：expected running time (续)

- 你能解释这里用的是哪种方法吗？

```
RandomSelect(A,i,n)
(selects the ith smallest element in set A, where n = |A| )
if (n = 1)
     return the one item in A
else
     p = randomElement(A)
     Let H be the set of elements greater than p
     Let L be the set of elements less than or equal to p
     If (H is empty)
          put p in H
     if (i ≤ |L|)
          Return RandomSelect(L, i, |L|)
     else
          Return RandomSelect(H, i − |L|, |H|).
```

$$T(n) \le \begin{cases} \frac{1}{2}T(\frac{3}{4}n) + \frac{1}{2}T(n) + bn & \text{if } n > 1 \\ d & \text{if } n = 1 \end{cases}$$

问题2： expected running time (续)

- 你能解释这里用的是哪种方法吗？

Exercise 5.6-4 Consider an algorithm that, given a list of n numbers, prints them all out. Then it picks a random integer between 1 and 3. If the number is 1 or 2, it stops. If the number is 3 it starts again from the beginning. What is the expected running time of this algorithm?

$$T(n) = \frac{2}{3}cn + \frac{1}{3}(cn + T(n))$$

问题2： expected running time (续)

- 你怎么理解indicator random variable？
- 怎么利用indicator random variable来简化期望的计算？

- 在这些问题中，indicator random variable分别可以是什么？
  - The expected number of times that we hire a new office assistant.
  - The expected number of pairs of people with the same birthday.
  - How many sixes do we expect to see on top if we roll 24 dice?（上周，根据期望的定义，我们是如何计算的？）

问题2：expected running time (续)

- 你怎么理解indicator random variable？
- 怎么利用indicator random variable来简化期望的计算？

$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}[X_i]$$

Given a sample space S and an event A in the sample space S, let $X_A = \mathrm{I}\{A\}$. Then $\mathrm{E}[X_A] = \Pr\{A\}$.

- 在这些问题中，indicator random variable分别可以是什么？
  - The expected number of times that we hire a new office assistant.
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问题2：expected running time (续)

- Suppose that you want to output 0 with probability $1/2$ and 1 with probability $1/2$. At your disposal is a procedure BIASED-RANDOM, that outputs either 0 or 1. It outputs 1 with some probability p and 0 with probability $1-p$, where $0 < p < 1$, but you do not know what p is. Give an algorithm that uses BIASED-RANDOM as a subroutine, and returns an unbiased answer, returning 0 with probability $1/2$ and 1 with probability $1/2$. What is the expected running time of your algorithm as a function of p?

问题2：expected running time (续)

- UNBIASED-RANDOM()

```
Output: 0 with probability 1/2 and 1 with probability 1/2
while true do
    a ← BIASED-RANDOM()
    b ← BIASED-RANDOM()
    if a < b then return 0
    if a > b then return 1
```

The algorithm calls BIASED-RANDOM twice to get two random numbers A and B. It repeats this until $A \neq B$. Then, depending on whether $A < B$ (that is, $A = 0$ and $B = 1$) or $A > B$ (that is, $A = 1$ and $B = 0$) it returns 0 or 1 respectively.

In any iteration, we have $\Pr(A < B) = p(1-p) = \Pr(B < A)$, that is, the probability that the algorithm returns 0 in that iteration equals to the probability that it returns 1 in that iteration. Since with probability 1 we return something at some point (and not repeat the loop endlessly) and the probabilities of returning 0 and 1 are equal in each iteration, the total probabilities of returning 0 and 1 must be $1/2$ and $1/2$ respectively.

- 怎么计算expected running time？

问题2：expected running time (续)

- UNBIASED-RANDOM()

  Output: 0 with probability 1/2 and 1 with probability 1/2

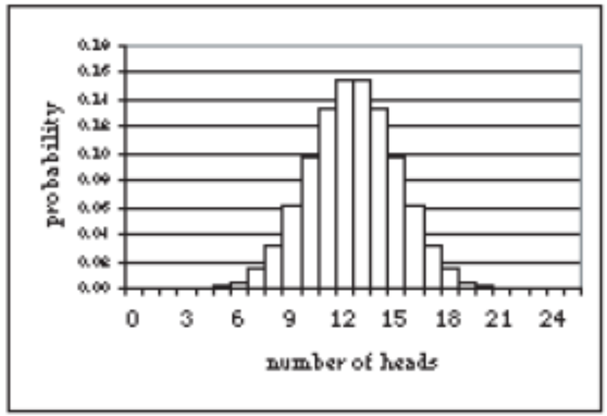
  while *true* **do**
      $a \leftarrow$ BIASED-RANDOM()
      $b \leftarrow$ BIASED-RANDOM()
      if $a < b$ **then return** 0
      if $a > b$ **then return** 1

- 怎么计算expected running time？

The algorithms stops, if it either returns 0 or 1. In every iteration, the probability of this is $\Pr(A \neq B) = \Pr(A < B) + \Pr(B < A) = 2p(1-p)$. Thus, we have a sequence of independent Bernoulli trials, each with probability $2p(1-p)$ of success. Therefore, the number of iterations required before the algorithm stops is geometrically distributed with parameter $2p(1-p)$, and the expected number of iterations is $1/(2p(1-p))$. As each iteration takes constant time (assuming that BIASED-RANDOM takes constant time), the expected running time of the algorithm is $\Theta(1/(p(1-p)))$.

问题3： probability distributions and variance

- 你怎么理解distribution function和它的histogram？



- 你怎么理解cumulative distribution function？
  它有哪些性质？
  什么情况下只能使用cumulative distribution function？

问题3： probability distributions and variance (续)

- 谈谈你对variance的理解
- 为什么variance被定义成$E((X-E(X))^2)$这种形式？