4-3 Dihedral Groups (D_n)

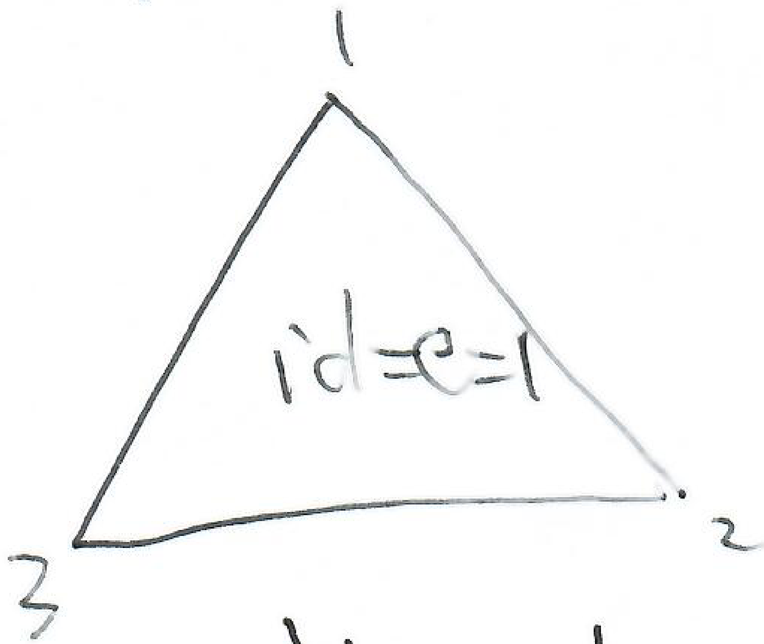
Def. D_n is the Group of Symmetries of regular n-gon.

EX. D_3.

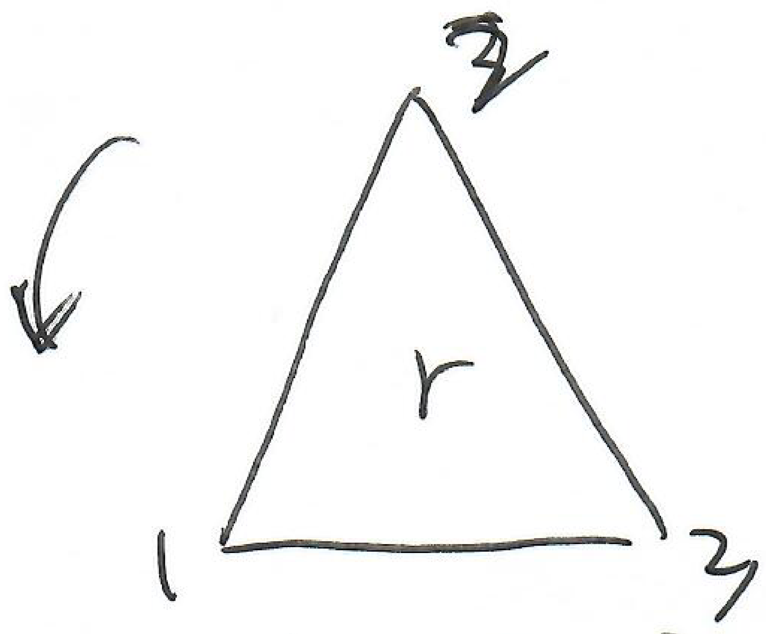
Outline

1. Elements of D_n. ~~Def and~~ ~~of D_n and their Orders~~.
2. Properties of Rotations & Reflections ~~Rotation~~ Between ~~Elements~~ (ch5-36)
3. Cayley digrams for D_n.
4. $C(D_n)$: Center of D_n (ch5-29)
5. D_n as directed products. (ch9-19) $D_6 \cong D_3 \times Z_2$.
   (internal / external.) (Conjecture & Prove) Thm 9.2)
6. ~~N~~ Subgroups of D_n
7. Normal Subgraphs of D_n
   and Quotient Groups of D_n.

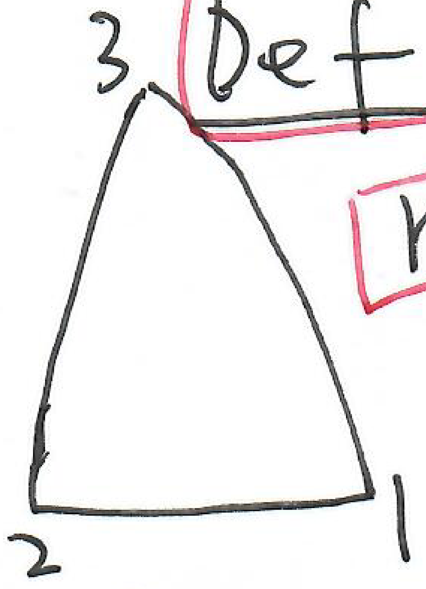
D_3: $n=3$ for an example. (D_4 Example 5.24).



$id=e=1$



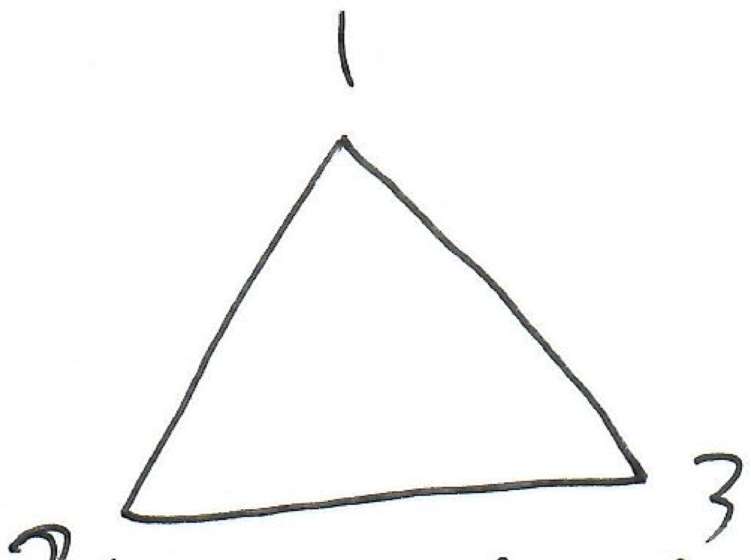
$r=(1\ 2\ 3)$



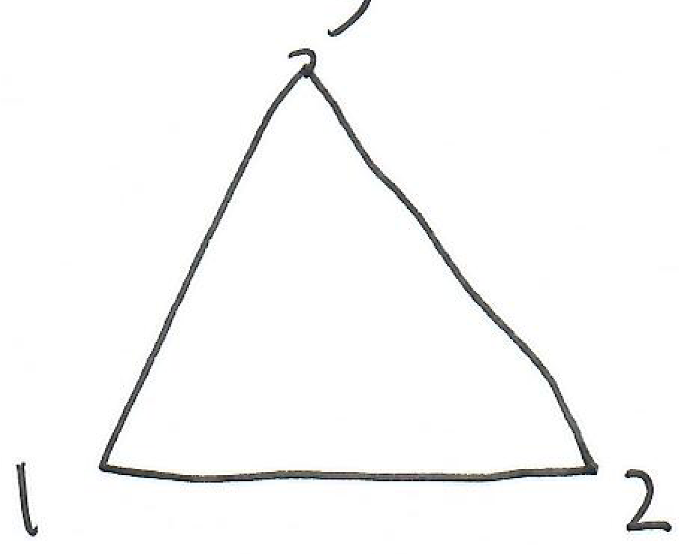
$r^2=(1\ 3\ 2)$.

$\boxed{\text{Def}}$

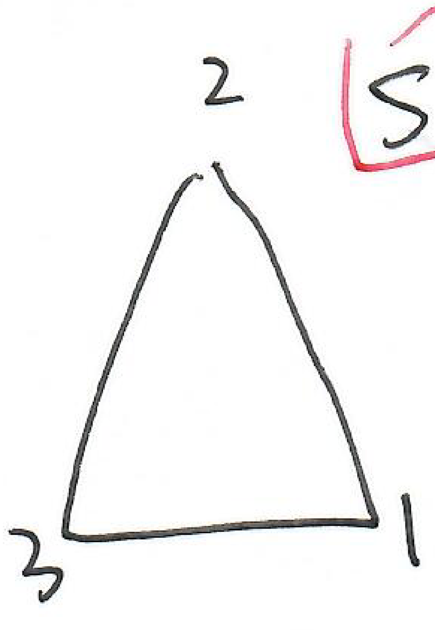
$\boxed{r}$: counterclockwise rotation by $\frac{2\pi}{n}$ radians.



$s=s_1=(2\ 3)$



$s_2=(1\ 3)$



$s_3=(1\ 2)$.

$\boxed{s}$: reflection across a line through vertex!

Thm $|D_n| = 2n \quad (n \geq 3)$

Pf. (1) $|D_n| \leq 2n$

(2) $|D_n| \geq 2n$.

Find 2n different elements of D_n.

$$D_n = \{1, r, r^2, \ldots r^{n-1}, \; s, rs, r^2s, \ldots, r^{n-1}s\}.$$

$\Leftarrow$ rotations r^i

$\Leftarrow$ reflections r^is

$0 \leq i \leq n-1$.

OR: $D_n = \{1, r, r^2, \ldots r^{n-1}, \; s_1, s_2, \ldots s^{n-1}\}$.

~~Thm. Example. rs~~

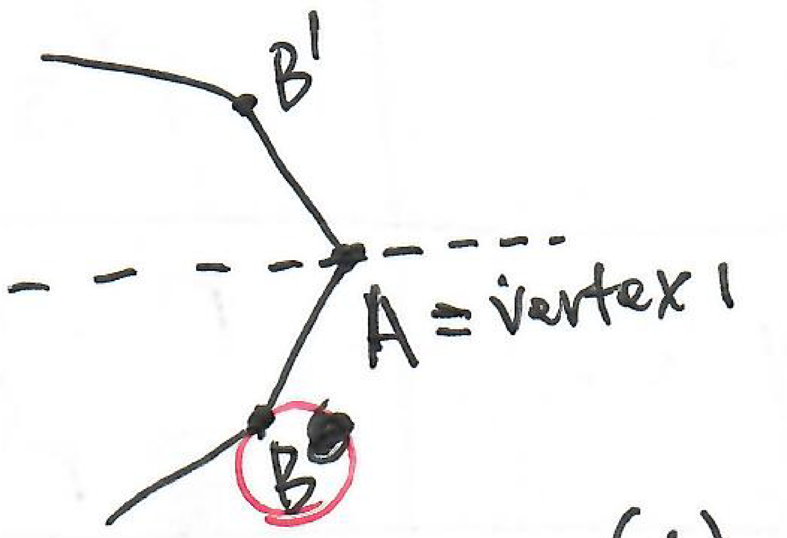
Thm (Orders)

(1) $|r^i| = \dfrac{n}{(i,n)}$.

(2) $|r^is| = 2$.

Pf for $srs^{-1} = r^{-1}$.

At some pair of adjacent vertices.



$srs^{-1}(A) = sr(A) = s(B') = B' \qquad r^{-1}(A) = B'$

$srs^{-1}(B) = sr(B') = s(A) = A \qquad r^{-1}(B) = A$.

Thm. (Relations Between Rotations and Reflections)

(1)

	Rot.	Ref.
Rot.	Rot.	Ref.
Ref.	Ref.	Rot

$\cong \mathbb{Z}_2 \cong D_n/\langle r \rangle$.

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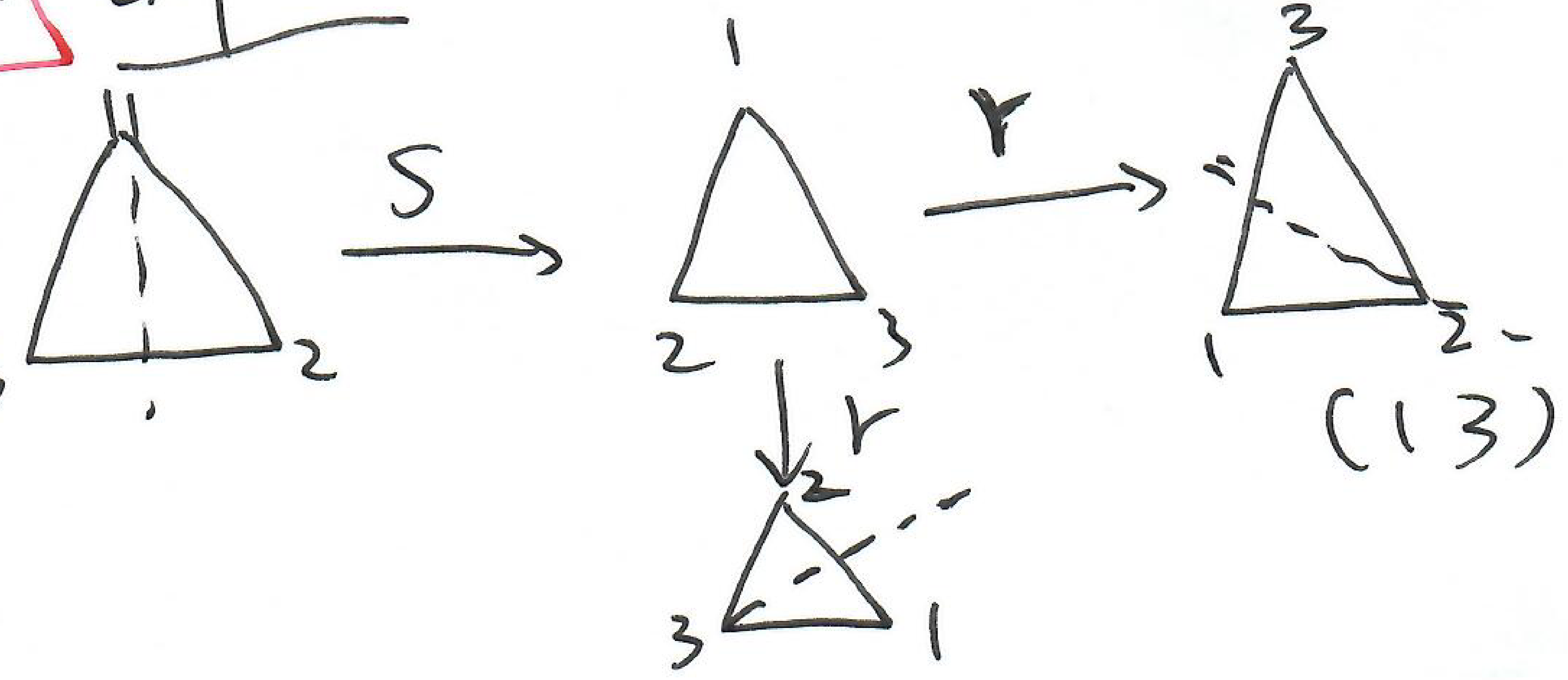
(2) $srs = r^{-1}$

$\Leftrightarrow \boxed{srs^{-1} = r^{-1}}$ "Conjugate".

$\Leftrightarrow \boxed{rs = sr^{-1}}$. Explain ~~Geometrically~~ in D_3.

$rs = (1\,2\,3)(2\,3)$

$= (1\,2)$



2 (3) $r^k s = s r^{\boxed{-k}}$.

Pf. $(srs^{-1})^k = (r^{-1})^k$

$\Rightarrow s r^k s^{-1} = r^{-k}$

$\Rightarrow r^k s = s r^{-k}$.

(4) $\forall a \in G$: a is of form $r^i s^j$ $(0 \le i \le n-1,\ 0 \le j \le 1)$.

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3. $Z(D_n)$: center of D_n. (ch5 - 29).

$Z(G) = \{ g \in G : gx = xg,\ \forall x \in G \}$.

$Z(G) \le G$. $Z(G) \lhd G$. (Q: $D_n / Z(G) \cong$?)

To find $C(D_n)$:

	$\forall.\ r^j$	$\forall.\ r^j s$
$\Rightarrow r^i$	Any i.	$r^i \cdot r^j s = r^j s r^i$ $\Rightarrow r^{i+j} s = r^{j-i} s$
$\Rightarrow r^i \cdot s$	Only $j = \frac{n}{2}$ ×	$\Rightarrow r^{i+j} = r^{j-i}$ $\Rightarrow (i+j)-(j-i) \equiv 0 \pmod n$ $2i \equiv 0 \pmod n$ $i = 0$ or $n/2$ (when n is even)

Thm: $n \ge 3$.

If n is odd, $C(D_n) = \{1\}$.

If n is even, $C(D_n) = \{1, r^{n/2}\}$.

Q: How to interpret $r^{n/2} \in D_n$ (n is even)?

3

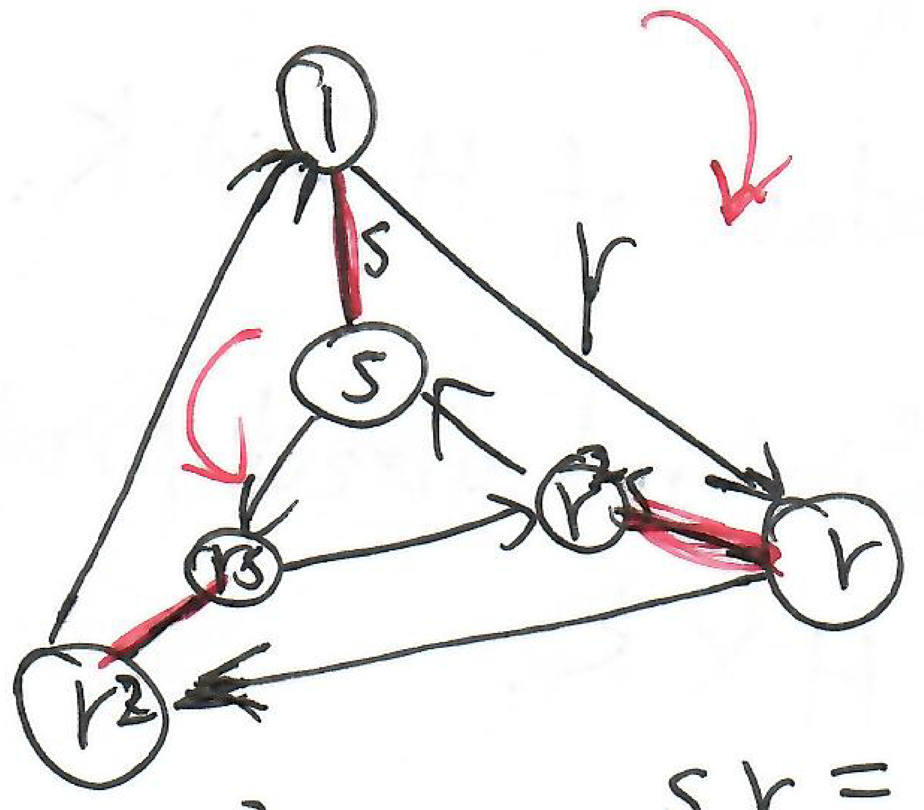
4. Cayley Diagrams for D_n.

Arthur Cayley (1821-1895)

|'keili|

$$D_n = \{1, r, r^2, \ldots, r^{n-1}, s, rs, r^2 s, \ldots, r^{n-1} s\}.$$

Cayley Diagrams for D_3:



$sr = r^{-1}s = r^2 s$

$sr^2 = r^{-2}s = rs$

D_4:

1

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5. D_n as direct product. (Ch 9-19).

(1) Prove that $S_3 \times \mathbb{Z}_2 \cong D_6$.

$D_6 \cong D_3 \times \mathbb{Z}_2$

(2) Conjecture about D_{2n}. Prove your conjecture.

Thm. If $n \geq 6$ is twice an odd number, then $D_n \cong D_{n/2} \times \mathbb{Z}_2$.

Note: $D_8 \not\cong D_4 \times \mathbb{Z}_2$

Pf. $|C(D_8) = \{1, r^4\}| = 2$

$|C(D_4) = \{1, r^2\}| = 2$

$|C(D_4 \times \mathbb{Z}_2)| = 2 \times 2 = 4.$

Pf. Let us first review Internal & External direct ~~prod~~ products.

④

External Direct Products

Def. $G = G_1 \times G_2$

$\{(g_1, g_2) : g_1 \in G_1,\ g_2 \in G_2\}$..

Internal Direct Products:

Def. $H \le G$, $K \le G$.

(1) $G = HK = \{hk : h \in H, k \in K\}$.

(2) $H \cap K = \{e\}$.

(3) $hk = kh$, $\forall h \in H, k \in K$.

Then G is the internal direct product of H and K.

$\Longleftrightarrow$ Def: $H \lhd G$, $K \lhd G$

(1) $G = HK$

(2) $H \cap K = \{e\}$.

Then G is the internal direct product of H and K.

$\boxed{\text{Thm 9.27}}$ G is the interal direct product of its subgroups H and K. Then $G \cong H \times K$.

Thm. (not used) If $G \cong H \times K$. Then ~~$\exists H' \lhd G, K' \lhd G$~~

$\exists H' \cong H \lhd G$, $K' \cong K \lhd G$

s.t. G is the interal direct product of H' and K'.

$D_n \cong D_{n/2} \times \mathbb{Z}_2$. ($n = 2k$ where $k = 2l+1$).

Pf. To find $H \cong D_{n/2}$, $K \cong \mathbb{Z}_2$ s.t G is the internal directed product of H and K

$\boxed{\text{s.t}}$ $H \lhd G$, $K \lhd G$,

(1) $G = HK$

(2) $H \cap K = \{e\}$.

$D_n \cong D_{n/2} \times \mathbb{Z}_2$

$\langle r^2, s\rangle \cong D_{n/2}$, $\{1, r^{n/2}\} \cong \mathbb{Z}_2$

$D_3 \lhd D_6$, $\mathbb{Z}_2 \lhd D_6$.

For example: $D_6 \cong D_3 \times \mathbb{Z}_2$

$D_6 \cong D_3\ \mathbb{Z}_2$ (internal direct product).

$\{1, r, r^2, r^3, r^4, r^5, s, rs, r^2s, r^3s, r^4s, r^5s\}$

$\langle r^2, s\rangle \cong D_3$ (even)

$\{1, r^3\} \cong \mathbb{Z}_2$ (✓) odd (center)

~~$\{1, s\}$~~ (X) (Not normal).

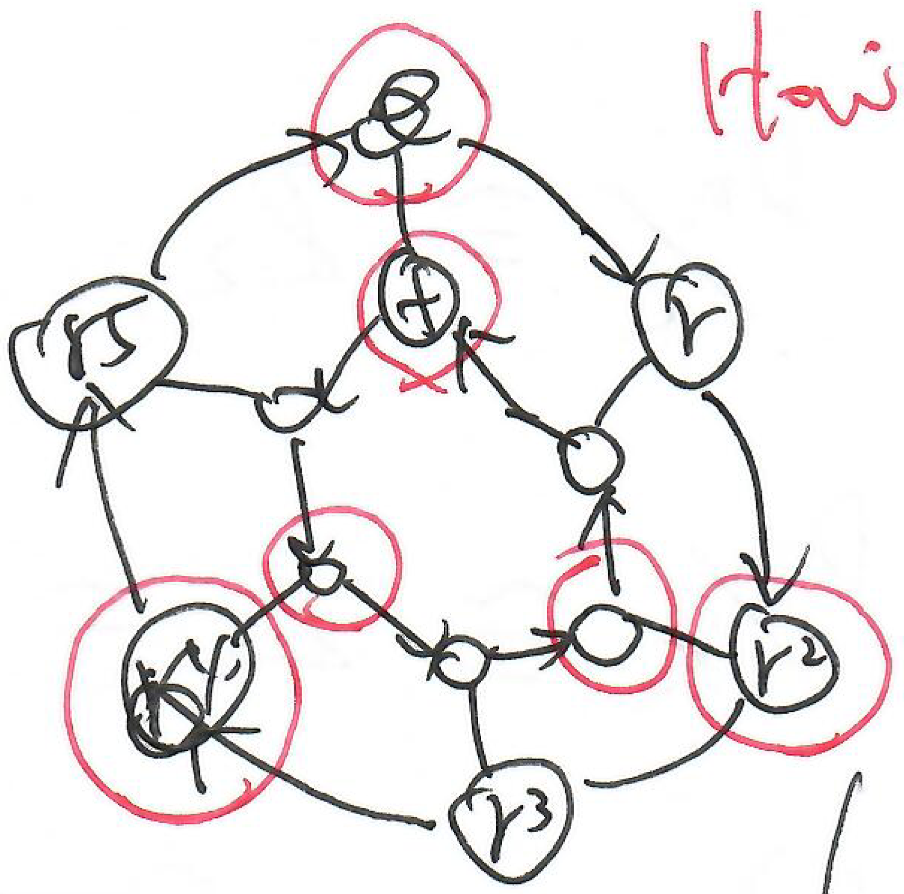
$\because$ $gNg^{-1} \neq N$ $(rsr^{-1} = r^2s \notin N)$.

5 Example for $D_n \cong D_{n/2} \times \mathbb{Z}_2$

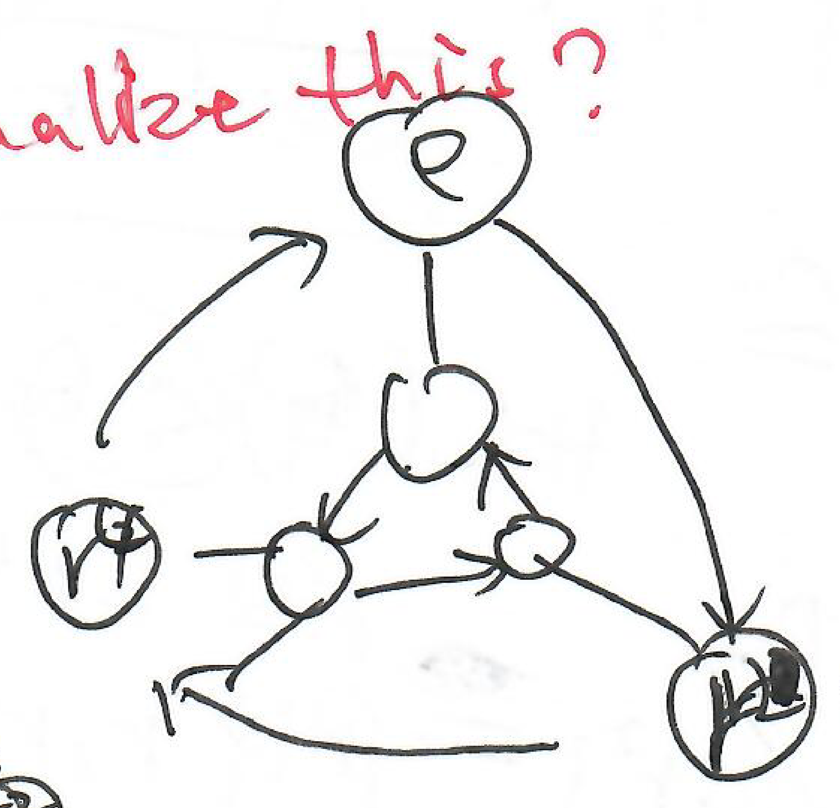
$\cancel{\ }\langle r^2, s\rangle \quad \{1, r^3\}.$

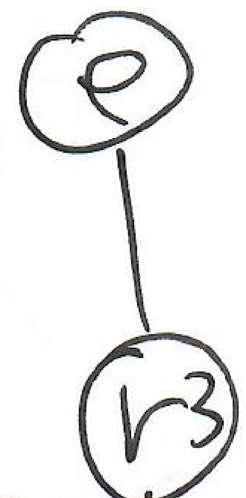
$D_6 \cong D_3 \times \mathbb{Z}_2$

How to visualize this?



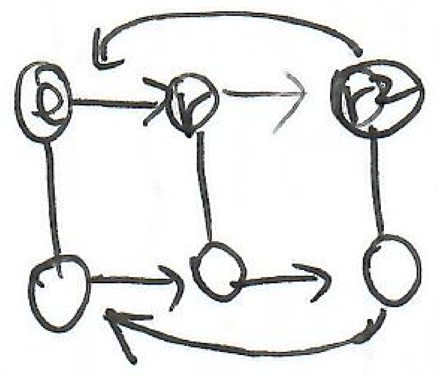
$\cong$





$\times \in$ for external

• for internal.



(D_3 organized by the subgroup $H = \langle r\rangle$).

Subgroups of D_n.

$D_n = \{1, r, r^2, \dots, r^{n-1},$
$s, rs, r^2s, \dots, r^{n-1}s\}.$

Q: Easy Proofs?

Thm. Every subgroup ~~of~~ of D_n is cyclic or dihedral.

(1) $\langle r^d\rangle$ where $d \mid n$. $\boxed{\cong}$ $\mathbb{Z}/(n/d)$ with index $2d$.

(2) $\langle r^d, r^i s\rangle$ where $d \mid n$, $0 \le i \le d-1$. $\boxed{\cong}$ $D_{n/d}$. with index $\underline{\underline{d}}$.

Analysis (Not a Proof.) D_6

$D_2, \quad D_4, \quad D_6.$

D_2: $|D_2| = 4$ $\quad D_2 = \{1, r, s, rs\}.$

$|H| = 1$: $\{1\}$

$|H| = 4$: $H = D_2$

$|H| = 2$: $\{1, r\}, \{1, s\}, \{1, rs\}.$

$|D_3| = 6$

$1, 2, 3, 6$

$|H| = 3$, $H \le D_3$.

~~$\{\langle r^2, s\rangle$ OR $\langle r^2, rs\rangle$~~

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(1) $\langle r^d\rangle$

(2) $\{1, \; r^i s\}.$

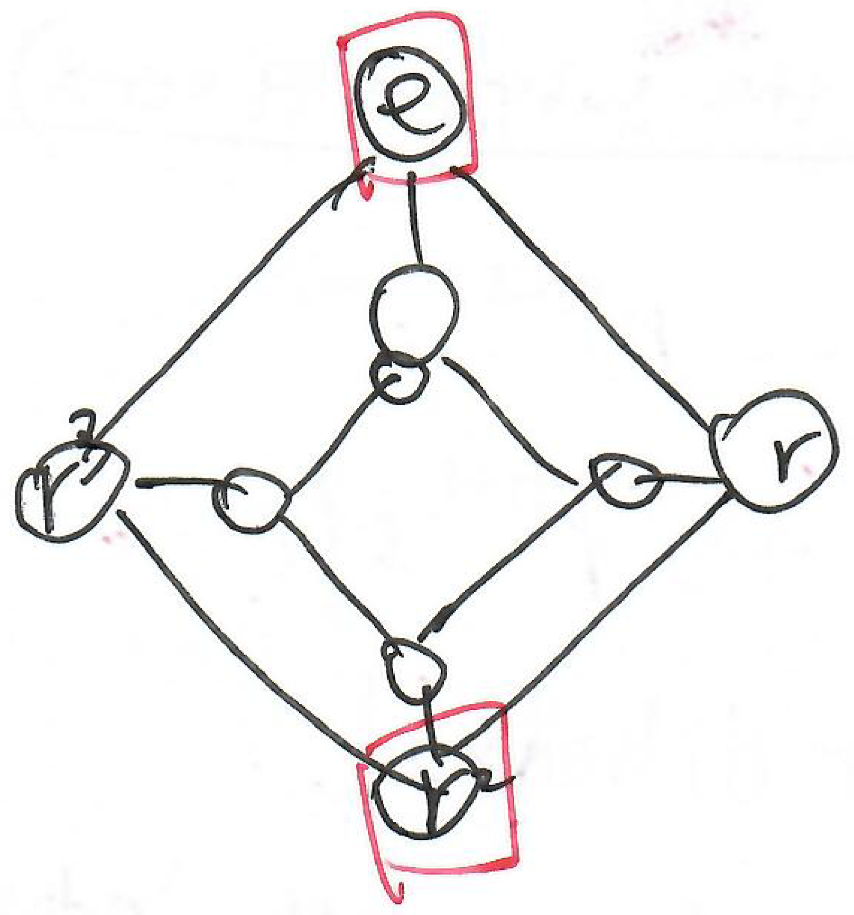
Normal Subgroups of D_n and Quotient Groups of D_n.

(6) $H=\{1, r^{n/2}\}$ when n is even. $\boxed{\text{Center}}$

$D_n/H \cong$

$H = C(G) \trianglelefteq G$

$$D_4/H \cong D_4/\{1,r^2\} \cong \left\{\begin{array}{l} H=\{1,r^2\}, \\ rH=\{r,r^3\} \\ sH=\{s,r^2s\} \\ rsH=\{rs,r^3s\} \end{array}\right\}$$



$\cong \{1,r,s,rs\}$.

四元群 Klein-group (4-group).

No proof! $\boxed{\text{Generally, } G/Z(G) \cong \mathrm{Inn}(G).}$

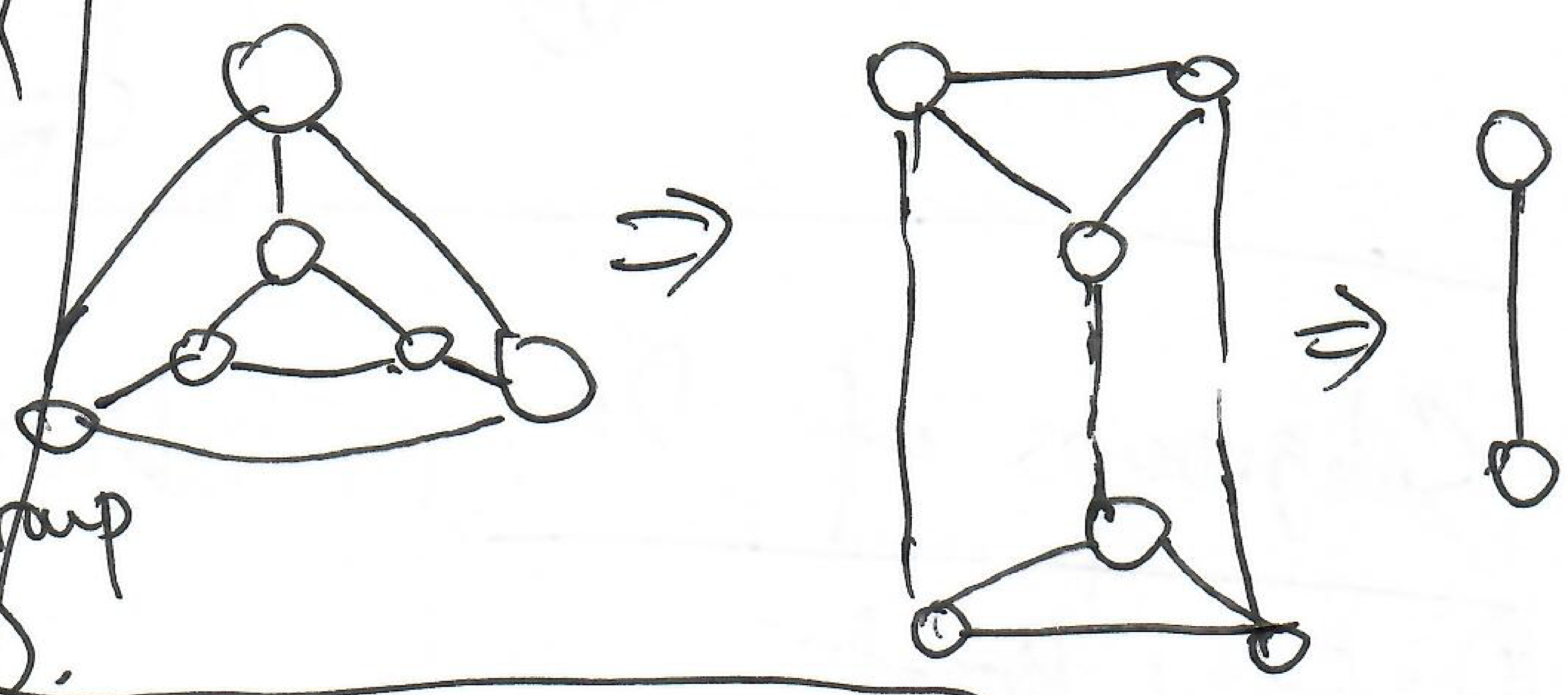
$\Downarrow$

$D_4=\{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$.

$K=\langle r\rangle$.

$D_n/K \cong \mathbb{Z}_2$.

$D_3/K \cong \mathbb{Z}_2 = \{1\cdot K, sK\}$



Only $\mathbb{Z}_2$: ~~rotation~~ reflection or Not.

	1	s
1	1	s
s	s	1

Two normal groups to study: when n is even.

$D_n/D_{n/2}$, $D_n/\{1,r^{n/2}\}$.

$D_n/\langle r\rangle \cong \mathbb{Z}_2$.