计算机问题求解 – 论题2-4
- 递归及其数学基础

2019年03月18日

Part I
递归与归纳

问题1：

书上是以什么方式引入数学归纳法的，你认为为什么以这样的方式引入？

用反证法证明：

(Euclid's Division Theorem, Restricted Version) Let n be a positive integer. Then for every nonnegative integer m, there exist unique integers q and r such that $m = nq + r$ and $0 \leq r < n$.

这需要什么背景条件?

For the purpose of a proof by contradiction, we assumed that there is a nonnegative integer m for which no such q and r exist. We chose a smallest such m and observed that $m - n$ is a nonnegative integer less than m. Then we said:

> Therefore, there exist integers q' and r' such that $m - n = nq' + r'$ with $0 \le r < n$. But then $m = n(q' + 1) + r'$. So, by taking $q = q' + 1$ and $r = r'$, we obtain $m = qn + r$ with $0 \le r < n$. This contradicts the assumption that there are no integers q and r with $0 \le r < n$ such that $m = qn + r$. Thus, by the principle of proof by contradiction, such integers q and r exist.

关键是：$p(m\text{-}n) \Rightarrow p(m)$

顺便问一句：如果只看上面一段，这个证明有什么问题?

Pivotal Role of the Proof

- We assumed that a counterexample with a smallest m existed.[1]
- Using the fact that $p(m')$ had to be true for every m' smaller than m, we chose $m' = m - n$ and observed that $p(m')$ had to be true.
- We used the implication $p(m - n) \Rightarrow p(m)$ to conclude the truth of $p(m)$.
- However, we had assumed that $p(m)$ was false, so this assumption is contradicted in the proof by contradiction.

反证法的“壳”；归纳法的“芯”

```
ProveSum(n)

// Assume that n is a positive integer.
// This is a recursive program that inputs n and prints a detailed proof
// showing that s(n) = n*(n+1)/2.
(1)  if (n == 1)
(2)      print "We note that"
(3)      print "  s(1) = 1 = 1*2/2, so the formula is correct for n = 1."
(4)  else
(5)      print "To prove that s(", n, ") = ", n , "*", n+1,
           "/2, we first prove that"
(6)      print "  s(", n-1, ") = ", n-1, "*", n, "/2."
(7)      proveSum(n-1)
(8)      print "Having proved s(", n-1,  ") = ", n-1, "*", n, "/2 = ",
           (n-1)*n/2," we add ", n
(9)      print "  to the first and last values, getting ",
           "s(", n, ") = ", ((n-1)*n/2 + n), "."
(10)     print "  This equals ", n, "*", n+1,
           "/2, so the formula is correct for n = " , n, "."
```

问题2：

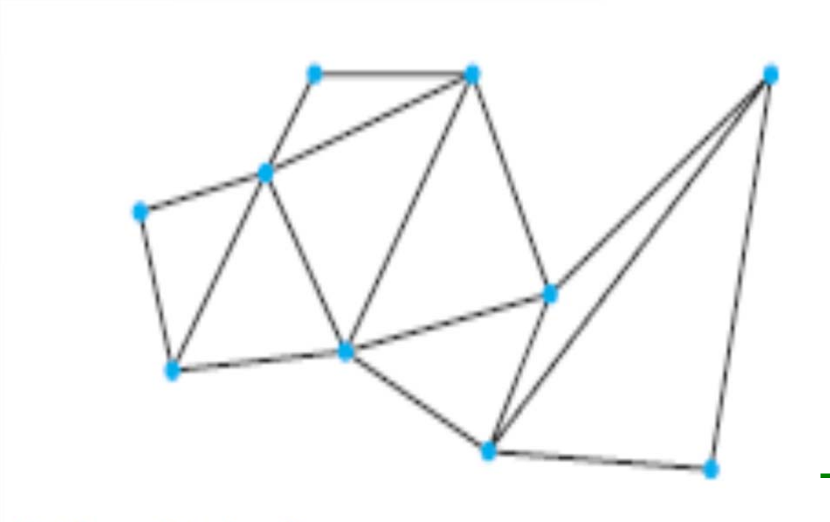
书上用这个递归过程想说明什么？

Because recursion works, we can call this program to print a proof for any *n*. Because a program exists that can generate a complete proof for any *n*, the property must be true for all *n*.

问题3：

这是什么意思？

结构归纳法



A triangulated polygon

问题4：

你能否简述一下如何用结构归纳法证明Ear Lemma？就此说明什么是结构归纳法？

一个三角形有3个耳朵；更大的凸多边形至少有2个在原多边形中不相邻的耳朵。

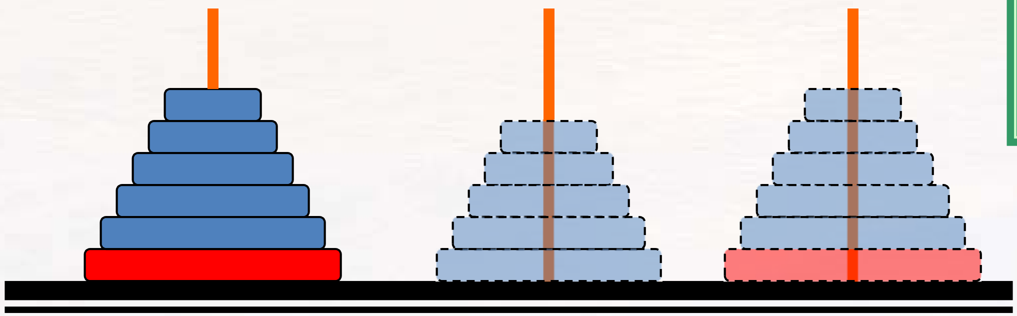
- 什么是“耳朵”？
- 什么是base case?
- 如何得到可以做“归纳假设”的子图？
- 如何从归纳假设得到结论？关键是合并子图后为什么原来的两个子图中的耳朵各至少留下1各，且不相邻。

问题5：

结构归纳法与你原来熟悉的针对自然数的归纳法有没有本质的不同？

递归用于计数

- Towers of Hanoi
  - How many moves are need to move all the disks to the third peg by moving only one at a time and never placing a disk on top of a smaller one.



$$T(1) = 1$$
$$T(n) = 2T(n-1) + 1$$

Solution of Towers of Hanoi

$T(n) = 2T(n-1) + 1$

$2T(n-1) = 4T(n-2) + 2$

$4T(n-2) = 8T(n-3) + 4$

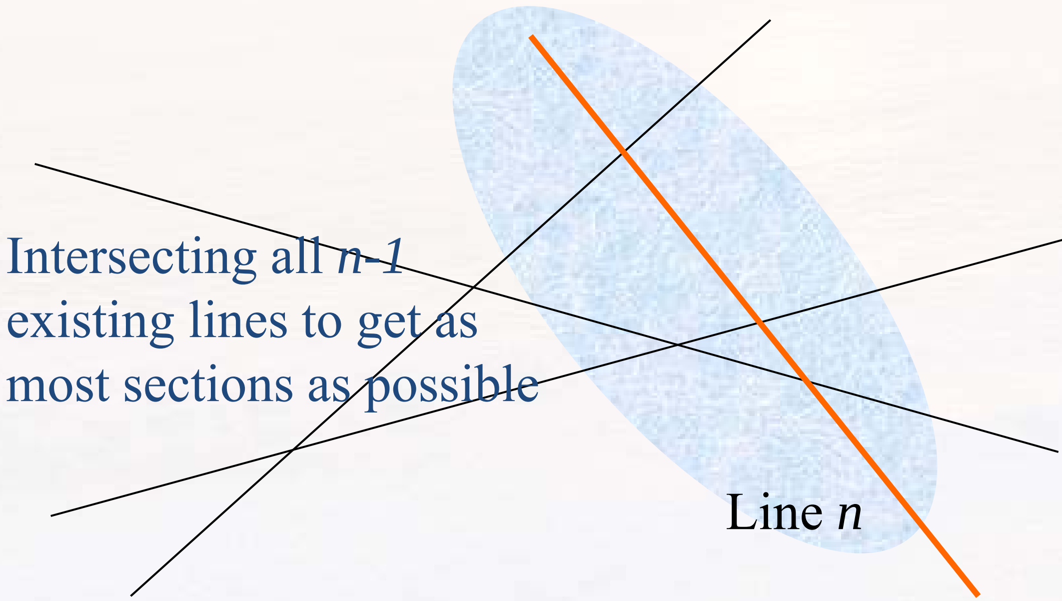
…….

$2^{n-2}T(2) = 2^{n-1}T(1) + 2^{n-2}$

$\Rightarrow$ $\boldsymbol{T(n) = 2^n - 1}$

递归用于计数：你试试

- Cutting the plane
  - How many sections can be generated at most by *n* straight lines with infinite length.



$L(0) = 1$

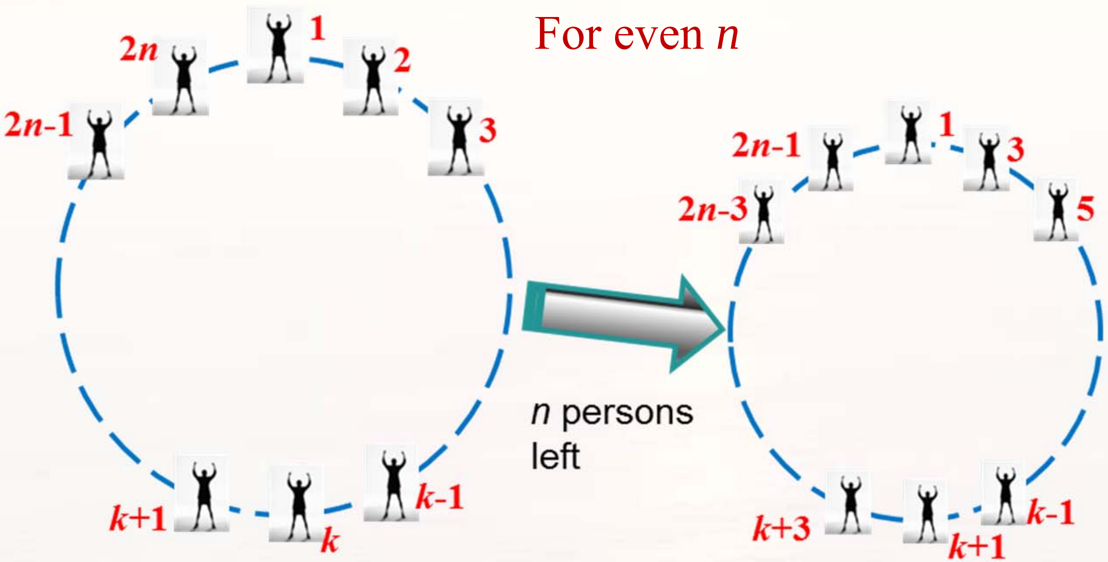
$L(n) = L(n-1) + n$

Solution of Cutting the Plane

$$L(n) = L(n-1)+n$$
$$= L(n-2)+(n-1)+n$$
$$= L(n-3)+(n-2)+(n-1)+n$$
$$= \ldots\ldots$$
$$= L(0)+1+2+\ldots\ldots+(n-2)+(n-1)+n$$

$$L(n) = n(n+1)/2 + 1$$

Josephus Problem



For even n

What about odd n?

The first person was killed, so, the new number would be ***k*+1**

The solution is: newnumber (J(n))

And the newnumber(k) is $2k-1$

Think the case of 5 persons, and 10 persons.

$$J(1) = 1;$$
$$J(2n) = 2J(n) - 1, \quad \text{for } n \geqslant 1;$$
$$J(2n+1) = 2J(n) + 1, \quad \text{for } n \geqslant 1.$$

Solution in Recursive Equations

$$J(1) = 1;$$

$$J(2n) = 2J(n) - 1, \qquad \text{for } n \geq 1;$$

$$J(2n+1) = 2J(n) + 1, \qquad \text{for } n \geq 1.$$

Explicit Solution for small *n*'s

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
J(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Look carefully ...

and, find the pattern...

and, prove it!

Eureka!

If we write *n* in the form $n = 2^m + l$,

(where 2^m is the largest power of 2 not exceeding *n* and where *l* is what's left),

the solution to our recurrence seems to be:

$$J(2^m + l) = 2l + 1, \qquad \text{for } m \geq 0 \text{ and } 0 \leq l < 2^m.$$

As an example: J(100) = J(64+36) = 36*2+1 = 73

Binary Representation

- Suppose *n*'s binary expansion is :

$$n = (b_m b_{m-1} \ldots b_1 b_0)_2$$

- then:

$$\begin{aligned} n &= (1\, b_{m-1}\, b_{m-2} \ldots b_1\, b_0)_2\,, \\ l &= (0\, b_{m-1}\, b_{m-2} \ldots b_1\, b_0)_2\,, \\ 2l &= (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, 0)_2\,, \\ 2l+1 &= (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, 1)_2\,, \\ J(n) &= (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, b_m)_2 \end{aligned}$$

$100 = 1100100_2$ → $1001001_2 = 73$

Part II
解递归以及分治法的代价

线性齐次递归式

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_m a_{n-k}$$

称为 k 次线性递归式

$$c_n = (-2)c_{n-1}$$

$$f_n = f_{n-1} + f_{n-2}$$

Yes

$$a_n = a_{n-1} + 3$$

$$g_n = g_{n-1}^2 + g_{n-2}$$

No

线性齐次递归式的特征方程

- 对于k次线性齐次递归式

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

下面的方程称为其特征方程：

$$x^k = r_1 x^{k-1} + r_2 x^{k-2} + \cdots + r_k$$

- 例如：二次线性齐次递归式的特征方程是: $x^2 - r_1 x - r_2 = 0$

问题6：

$$a_n = u s_1^n + v s_2^n$$

你能说出一个熟悉的二次线性齐次递归式吗？

$f_1=1$

$f_2=1$

$f_n=f_{n-1}+f_{n-2}$

→ **1, 1, 2, 3, 5, 8, 13, 21, 34, ...**

问题7:

这是Fibonacci序列，你知道它为什么那么出名吗？

$$\frac{F_{n+1}}{F_n}$$

$$\frac{1}{1} = 1.000000000$$

$$\frac{2}{1} = 2.000000000$$

$$\frac{3}{2} = 1.500000000$$

$$\frac{5}{3} = 1.666666667$$

$$\frac{8}{5} = 1.600000000$$

$$\frac{13}{8} = 1.625000000$$

$$\frac{21}{13} = 1.615384615$$

$$\frac{34}{21} = 1.619047619$$

$$\frac{55}{34} = 1.617647059$$

$$\frac{89}{55} = 1.6182181618$$

$$\frac{144}{89} = 1.617977528$$

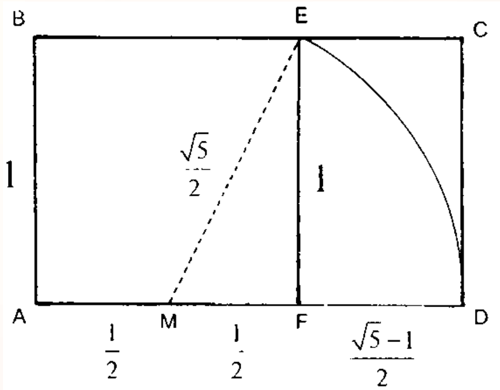
$$\frac{233}{144} = 1.618055556$$

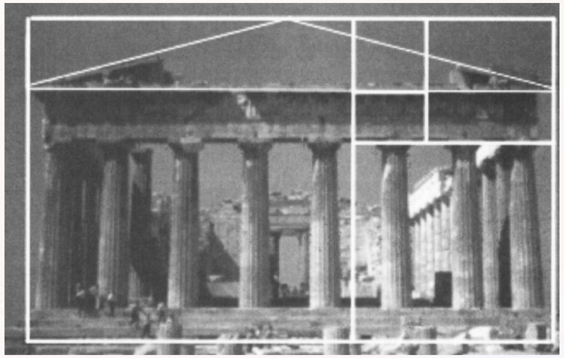
$$\frac{377}{233} = 1.618025751$$

$$\frac{610}{377} = 1.618037135$$

$$\frac{987}{610} = 1.618032787$$

黃金分割





PHIDIAS (460-430 BC)
OR
φ-BONACCI ?

Fibonacci 序列

$f_1=1$

$f_2=1$

$f_n= f_{n-1}+ f_{n-2}$

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

费波纳齐序列的显式公式：

其特征方程 $x^2-x-1=0$, 有两个实根:

$$s_1=\frac{1+\sqrt{5}}{2} \quad and \quad s_2=\frac{1-\sqrt{5}}{2}$$

根据初始条件解待定系数：$f_1=us_1+vs_2=1 \;\; and \;\; f_2=us_1^{\,2}+vs_2^{\,2}=1$

结果是:

$$f_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

分治法导出的递归式

最简单的形式：

$$T(n) = aT\left(\frac{n}{2}\right) + n,$$

1. If $a < 2$, then $T(n) = \Theta(n)$.
2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.

问题8：

你能解释一下三种情况的差别及其解的背景吗？

Suppose that we have a recurrence of the form

$$T(n) = aT\left(\frac{n}{2}\right) + n,$$

where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

1. If $a < 2$, then $T(n) = \Theta(n)$.
2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.

问题9：

你能解释一下这是如何“推广”的吗？

(Master Theorem, Preliminary Version) Let a be an integer greater than or equal to 1, and let b be a real number greater than 1. Let c be a positive real number, and d, a nonnegative real number. Given a recurrence of the form

$$T(n) = \begin{cases} aT(n/b) + n^c & \text{if } n > 1, \\ d & \text{if } n = 1, \end{cases}$$

in which n is restricted to be a power of b, we get the following:

1. If $\log_b a < c$, then $T(n) = \Theta(n^c)$.
2. If $\log_b a = c$, then $T(n) = \Theta(n^c \log n)$.
3. If $\log_b a > c$, then $T(n) = \Theta(n^{\log_b a})$.

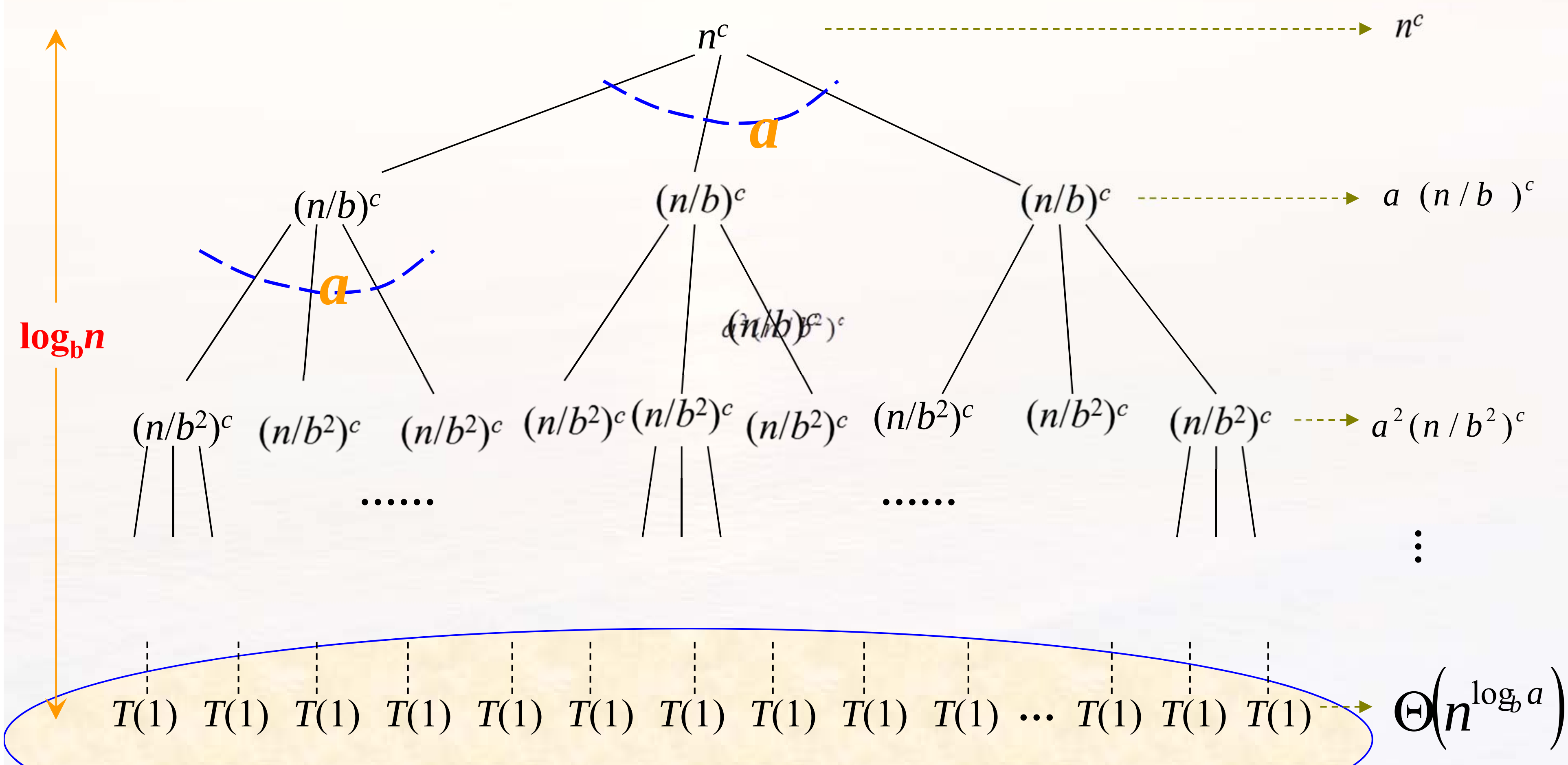
证明的关键

$$\log_b a \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} c \text{ iff. } \left(\frac{a}{b^c}\right) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 1$$

利用递归树容易得到，各层代价之和为：

$$n^c \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i$$

Recursion Tree for $T(n)=aT(n/b)+n^c$



Note: $a^{\log_b n} = n^{\log_b a}$

Total ?

问题10：

上述Master Theorem的局限性在何处？我们如何进一步推广？

$$T(n) = \begin{cases} 2T(n/3) + 4n^{3/2} & \text{if } n > 1, \\ d & \text{if } n = 1, \end{cases}$$

⇒

$$T'(n) = \begin{cases} 2T'(n/3) + n^{3/2} & \text{if } n > 1, \\ d & \text{if } n = 1. \end{cases}$$

$$S(n) = \begin{cases} 2S(n/3) + f(n) & \text{if } n > 1, \\ d & \text{if } n = 1, \end{cases}$$

$$f(n) = n\sqrt{n+1}$$

⇒

$$n\sqrt{n+1} > n\sqrt{n} = n^{3/2}$$

$$n\sqrt{n+1} \le 4n^{3/2} \quad \text{for} \quad n \ge 0.$$

$$S(n) = \Theta(T'(n))$$

(Master Theorem) Let a and b be positive real numbers, with $a \geq 1$ and $b > 1$. Let $T(n)$ be defined for integers n that are powers of b by

$$T(n) = \begin{cases} aT(n/b) + f(n) & \text{if } n > 1, \\ d & \text{if } n = 1. \end{cases}$$

Then we have the following:

1. If $f(n) = \Theta(n^c)$, where $\log_b a < c$, then $T(n) = \Theta(n^c) = \Theta(f(n))$.
2. If $f(n) = \Theta(n^c)$, where $\log_b a = c$, then $T(n) = \Theta(n^c \log n) = \Theta(f(n) \log n)$.
3. If $f(n) = \Theta(n^c)$, where $\log_b a > c$, then $T(n) = \Theta(n^{\log_b a})$.

其实推广的步子不算大，但再推广证明就比较麻烦了！

课外作业

- CS pp.180-: prob. 16, 17
- CS pp.197-: prob. 8, 11, 17
- CS pp.212-: prob. 9, 13, 16
- CS pp.221-: prob. 1, 4, 6
- CS pp.233-: prob. 8-10