3-14 Planar & Coloring

Jun Ma

majun@nju.edu.cn

December 22, 2020

CZ 9.3

(a) The vertices of a certain graph G have degrees 3, 4, 4, 4, 5, 6, 6. Prove that G is nonplanar.

CZ 9.3

(a) The vertices of a certain graph G have degrees 3, 4, 4, 4, 5, 6, 6. Prove that G is nonplanar.

Proof.

$$n = 7, m = (3+4+4+4+5+6+6)/2 = 16$$

$$\Downarrow$$

$$3n - 6 = 15 < 16 = m$$

□

If G is a graph of order $n \geq 3$ and size m such that $m > 3n - 6$, then G is nonplanar.

CZ 9.3

(b) The vertices of a certain graph G have degrees 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7. Prove that G is nonplanar.

CZ 9.3

(b) The vertices of a certain graph G have degrees 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7. Prove that G is nonplanar.

Proof.

$$n = 12, m = 66/2 = 33$$

$$\Downarrow$$

$$3n - 6 = 30 < 33 = m$$

□

If G is a graph of order $n \geq 3$ and size m such that $m > 3n - 6$, then G is nonplanar.

CZ 9.5

Show that there exists

(a) a 4-regular planar graph and a 4-regular nonplanar graph.

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$$m = 4n/2 = 2n \text{ and } m \leq 3n - 6 \Rightarrow n \geq 6$$

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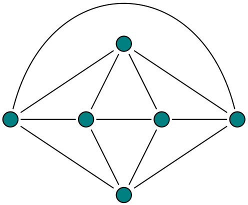

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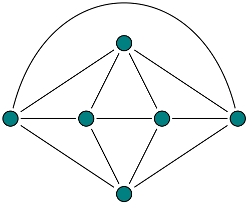

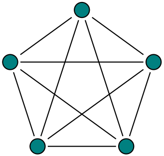

K_5

CZ 9.5

Show that there exists

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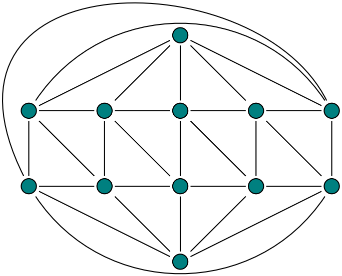

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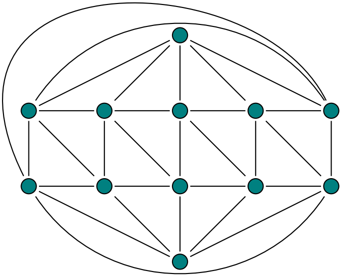

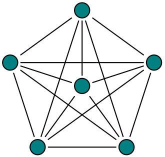

K_6

CZ 9.5

Show that there exists

(c) no r-regular planar graph for $r \geq 6$.

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$$(r \cdot n)/2 = m \leq 3n - 6$$

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Impossible!

CZ 9.7

Give an example of each of the following or explain why no such example exists.

(a) a planar graph of order 4.

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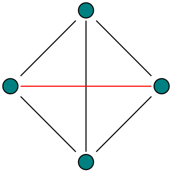

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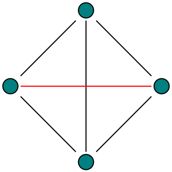

Is it right?

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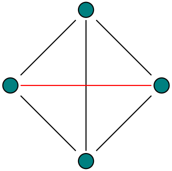

Is it right? ✓

CZ 9.7

Give an example of each of the following or explain why no such example exists.

(b) a nonplanar graph of order 4.

CZ 9.7

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Not Exists!

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Not Exists!

It cannot contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

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It cannot contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

Theorem 9.7 (Kuratowski's Theorem) *A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

CZ 9.7

Give an example of each of the following or explain why no such example exists.

(c) a nonplanar graph of **order 6** that contains neither K_5 nor $K_{3,3}$ as a subgraph.

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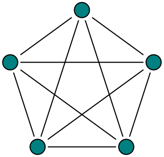

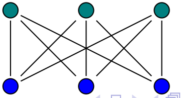

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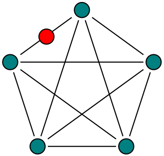

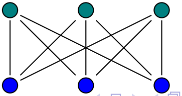

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Give an example of each of the following or explain why no such example exists.

(d) a plane graph having 5 vertices, 10 edges and 7 regions.

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Not Exists!

$$n - m + r = 5 - 10 + 7 = 2, \checkmark$$

CZ 9.7

Give an example of each of the following or explain why no such example exists.

(d) a plane graph having 5 vertices, 10 edges and 7 regions.

Not Exists!

$$n - m + r = 5 - 10 + 7 = 2, ✓$$

However

$$m = 10 > 9 = 3n - 6, ✗$$

CZ 9.7

Give an example of each of the following or explain why no such example exists.

(e) a planar graph of order $n \geq 3$ and size m with $m = 3n - 6$.

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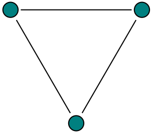
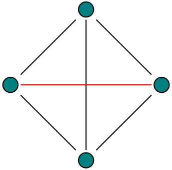


$n = 3, m = 3$



$n = 4, m = 6$

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Give an example of each of the following or explain why no such example exists.

(f) a **noplanar** graph of order $n \geq 3$ and size m with $m = 3n - 6$.

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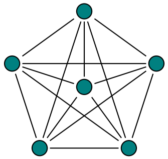

K_6

$$n = 6, m = 15 \Rightarrow m = 15 > 12 = 3n - 6$$

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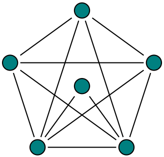

$$n = 6, m = 12 \Rightarrow m = 12 = 3n - 6$$

CZ 9.8

Determine, with explanation, whether the graph $K_4 \times K_2$ is planar.

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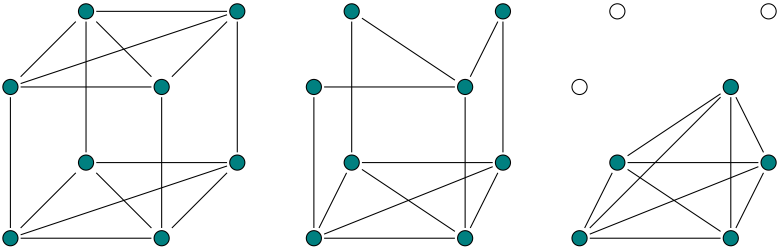

Theorem 9.7 (Kuratowski's Theorem) *A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

CZ 10.2

Determine the chromatic number of each of the following:

(a) the Petersen graph

(b) the n-cube Q_n

(c) the wheel $W_n = C_n + K_1$.

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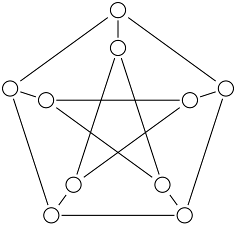

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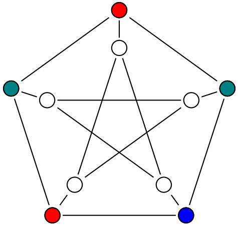

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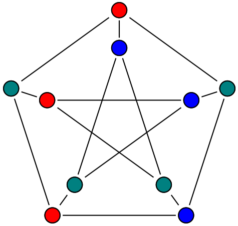

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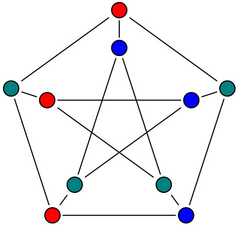

$$\chi(PG) = 3$$

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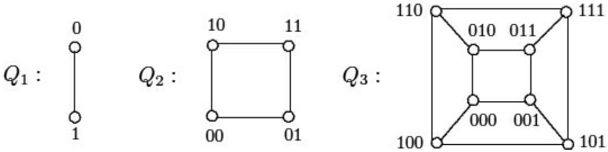

Figure 1.33: The n-cubes for $1 \leq n \leq 3$

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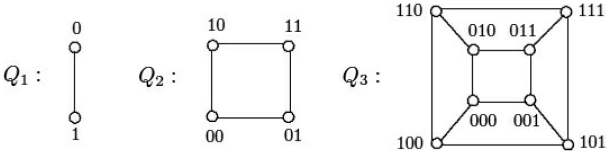

Figure 1.33: The n-cubes for $1 \le n \le 3$

$$\chi(Q_n) = 2$$

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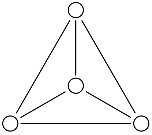

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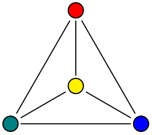

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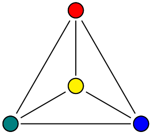

$\chi(W_3) = 4$

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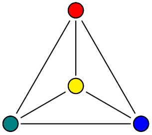

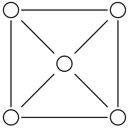

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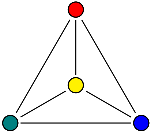

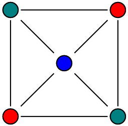

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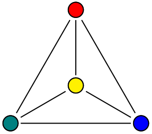

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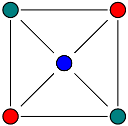

$\chi(W_4) = 3$

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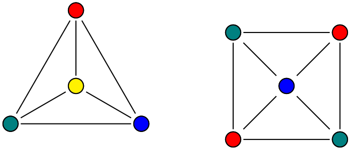

$\chi(W_3) = 4$ $\chi(W_4) = 3$

$$\chi(W_n) = \begin{cases} 4 & n = 2k+1 \\ 3 & n = 2k \end{cases}$$

CZ 10.3

What is the chromatic number of a tree?

CZ 10.3

What is the chromatic number of a tree?

$$\chi(T) = \begin{cases} 1 & |T.V| = 1 \\ 2 & \text{otherwise} \end{cases}$$

CZ 10.4

Prove or disprove:

(a) If a planar graph contains a triangle, then its chromatic number is 3.

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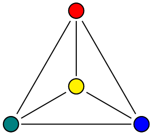

$$\chi(W_3) = 4$$

CZ 10.4

Prove or disprove:

(b) If there is a 4-coloring of a graph G, then $\chi(G) = 4$.

CZ 10.4

Prove or disprove:

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NO!

CZ 10.4

Prove or disprove:

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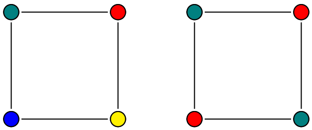
NO!



CZ 10.4

Prove or disprove:

(c) If it can be shown that there is no a 3-coloring of a graph G, then $\chi(G) = 4$.

CZ 10.4

Prove or disprove:

(c) If it can be shown that there is no a 3-coloring of a graph G, then $\chi(G) = 4$.

NO!

$$\chi(G) \geq 4$$

CZ 10.4

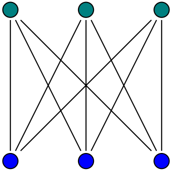
Prove or disprove:

(d) If G is a graph with $\chi(G) \leq 4$, then G is planar.

CZ 10.4

Prove or disprove:

(d) If G is a graph with $\chi(G) \leq 4$, then G is planar.



$K_{3,3}$

CZ 10.5

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- Assume k edges are removed from K_6, the chromatic number of the remaining graph G is

$$\chi(G) \geq \chi(K_6) - k$$

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- As $\chi(G) = 3$, so

$$3 \geq 6 - k \Rightarrow k \geq 3$$

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At least 3 edges need to be removed from K_6.

□


Thank
You!