- 教材讨论
  - JH第2章第3节

问题1：字母表、词、语言

- alphabet、symbol、word、language
  - 它们是如何被形式化定义的？
  - 你能举出一些实际生活中的例子吗？

- 你能设计一种语言来编码全班同学问求的期末考试成绩吗？
  - 你设计的字母表、词和语言分别是什么？
- 你能设计一种语言来编码图片吗？
  - 你设计的字母表、词和语言分别是什么？
- 编码视频呢？
  - 你设计的字母表、词和语言分别是什么？

- 什么是concatenation of word？
  你能利用这个概念来定义这些新概念吗？
  - prefix/suffix/subword
  - concatenation of language

问题1：字母表、词、语言 (续)

Definition 2.3.1.10. *Let $\Sigma = \{s_1, s_2, \ldots, s_m\}$, $m \geq 1$, be an alphabet, and let $s_1 < s_2 < \cdots < s_m$ be a linear ordering on Σ. We define the* **canonical ordering** *on Σ^* as follows. For all $u, v \in \Sigma^*$,*

$$
\begin{aligned}
u < v \ \text{if}\ & |u| < |v| \\
& \text{or}\ |u| = |v|, u = xs_iu',\ \text{and}\ v = xs_jv' \\
& \quad \text{for some}\ x, u', v' \in \Sigma^*,\ \text{and}\ i < j.
\end{aligned}
$$

- 我们为什么需要一种词的排序规则？
- 你能解释这条排序规则吗？
- 你能给出一种不同的排序规则吗？

问题2：判定和优化问题

Definition 2.3.2.1. *A* **decision problem** *is a triple (L, U, Σ) where Σ is an alphabet and $L \subseteq U \subseteq \Sigma^*$. An algorithm A* **solves (decides)** *the decision problem (L, U, Σ) if, for every $x \in U$,*

(i) $A(x) = 1$ if $x \in L$, and
(ii) $A(x) = 0$ if $x \in U - L$ ($x \notin L$).

- decision problem中的三个符号分别表示什么意思？
  - 这里的word是什么？
- 判定算法应该给出怎样的结果？

An equivalent form of a description of a decision problem is the following form that specifies the input-output behavior.

Problem (L, U, Σ)

Input: An $x \in U$.
Output: "yes" if $x \in L$,
"no" otherwise.

For many decision problems (L, U, Σ) we assume $U = \Sigma^*$. In that case we shall use the short notation (L, Σ) instead of (L, Σ^*, Σ).

- 你理解这段话的含义了吗？

问题2：判定和优化问题 (续)

- 这些判定问题分别是什么含义？它们的L分别是什么？
  - Primality testing $\{w \in \{0,1\}^* \mid Number(w) \text{ is a prime}\}$
  - Equivalence problem for polynomials
  - Satisfiability problem $\{w \in \Sigma_{logic}^{+} \mid w \text{ is a code of a satisfiable formula in CNF}\}$
  - Clique problem $\{x\#w \in \{0,1,\#\}^* \mid x \in \{0,1\}^* \text{ and } w \text{ represents a graph that contains a clique of size } Number(x)\}$
  - Vertex cover problem $\{u\#w \in \{0,1,\#\}^+ \mid u \in \{0,1\}^+ \text{ and } w \text{ represents a graph that contains a vertex cover of size } Number(u)\}$
  - Hamiltonian cycle problem $\{w \in \{0,1,\#\}^* \mid w \text{ represents a graph that contains a Hamiltonian cycle}\}$
  - Existence of a solution of linear integer programming $\{\langle A,b\rangle \in \{0,1,\#\}^* \mid Sol_{\mathbb{Z}}(A,b) \neq \emptyset\}$

问题2：判定和优化问题 (续)

Definition 2.3.2.2. *An* **optimization problem** *is a 7-tuple* $U = (\Sigma_I, \Sigma_O, L, L_I, \mathcal{M}, cost, goal)$, *where*

(i) Σ_I *is an alphabet, called the* **input alphabet** *of* U,
(ii) Σ_O *is an alphabet, called the* **output alphabet** *of* U,
(iii) $L \subseteq \Sigma_I^*$ *is the* **language of feasible problem instances**,
(iv) $L_I \subseteq L$ *is the* **language of the (actual) problem instances of** U,
(v) $\mathcal{M}$ *is a function from* L *to* $Pot(\Sigma_O^*)$,[30] *and, for every* $x \in L$, $\mathcal{M}(x)$ *is called the* **set of feasible solutions** *for* x,
(vi) $cost$ *is the* **cost function** *that, for every pair* (u, x), *where* $u \in \mathcal{M}(x)$ *for some* $x \in L$, *assigns a positive real number* $cost(u, x)$,
(vii) $goal \in \{minimum, maximum\}$.

- optimization problem中的七个符号分别表示什么意思？

An algorithm A *is* **consistent** *for* U *if, for every* $x \in L_I$, *the output* $A(x) \in \mathcal{M}(x)$. *We say that an algorithm* B **solves** *the optimization problem* U *if*

(i) B *is consistent for* U, *and*
(ii) for every $x \in L_I$, $B(x)$ *is an optimal solution for* x *and* U.

- 优化算法应该给出怎样的结果？

问题2：判定和优化问题 (续)

- 你能简述minimum vertex cover problem的含义吗？

Input: A graph $G = (V, E)$.
Constraints: $\mathcal{M}(G) = \{S \subseteq V \mid$ every edge of E is incident to at least one vertex of $S\}$.
Cost: For every $S \in \mathcal{M}(G)$, $cost(S, G) = |S|$.
Goal: *minimum*.

- 它和判定问题中的vertex cover problem之间有什么联系？

$$\{u\#w \in \{0,1,\#\}^+ \mid u \in \{0,1\}^+ \text{ and } w \text{ represents a graph that contains a vertex cover of size } Number(u)\}$$

问题2：判定和优化问题 (续)

- 你能简述maximum clique problem的含义吗？

Input: A graph $G = (V, E)$
Constraints: $\mathcal{M}(G) = \{S \subseteq V \mid \{\{u,v\} \mid u, v \in S, u \neq v\} \subseteq E\}$.
$\{\mathcal{M}(G)$ contains all complete subgraphs (cliques) of $G\}$
Costs: For every $S \in \mathcal{M}(G)$, $cost(S, G) = |S|$.
Goal: *maximum*.

- 它和判定问题中的clique problem之间有什么联系？

$\{x\#w \in \{0,1,\#\}^* \mid x \in \{0,1\}^*$ and w represents a graph that contains a clique of size $Number(x)\}$

问题2：判定和优化问题 (续)

- 你能简述maximum cut problem的含义吗？

Input: A graph $G = (V, E)$.
Constraints:
$$\mathcal{M}(G) = \{(V_1, V_2) \mid V_1 \cup V_2 = V, V_1 \neq \emptyset \neq V_2, \text{and} V_1 \cap V_2 = \emptyset\}.$$
Costs: For every cut $(V_1, V_2) \in \mathcal{M}(G)$,
$$cost((V_1, V_2), G) = |E \cap \{\{u, v\} \mid u \in V_1, v \in V_2\}|.$$
Goal: *maximum*.

- 你能给出一个与之相关的判定问题吗？

问题2：判定和优化问题 (续)

- 你能简述traveling salesperson problem的含义吗？

Input: A weighted complete graph (G, c), where $G = (V, E)$ and $c : E \rightarrow \mathbb{N}$. Let $V = \{v_1, \ldots, v_n\}$ for some $n \in \mathbb{N} - \{0\}$.

Constraints: For every input instance (G, c), $\mathcal{M}(G, c) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_n}, v_{i_1} \mid (i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)\}$, i.e., the set of all Hamiltonian cycles of G.

Costs: For every Hamiltonian cycle $H = v_{i_1} v_{i_2} \ldots v_{i_n} v_{i_1} \in \mathcal{M}(G, c)$, $cost((v_{i_1}, v_{i_2}, \ldots v_{i_n}, v_{i_1}), (G, c)) = \sum_{j=1}^{n} c(\{v_{i_j}, v_{i_{(j \bmod n)+1}}\})$, i.e., the cost of every Hamiltonian cycle H is the sum of the weights of all edges of H.

Goal: *minimum*.

问题2：判定和优化问题 (续)

- 你能简述knapsack problem的含义吗？

Input: A positive integer b, and $2n$ positive integers $w_1, w_2, \ldots, w_n$, c_1, $c_2, \ldots, c_n$ for some $n \in \mathbb{N} - \{0\}$.

Constraints:

$\mathcal{M}(b, w_1, \ldots, w_n, c_1, \ldots, c_n) = \{T \subseteq \{1, \ldots, n\} \mid \sum_{i \in T} w_i \leq b\}$.

Costs: For each $T \in \mathcal{M}(b, w_1, \ldots, w_n, c_1, \ldots, c_n)$,

$$cost(T, b, w_1, \ldots, w_n, c_1, \ldots, c_n) = \sum_{i \in T} c_i.$$

Goal: *maximum*.

问题2：判定和优化问题 (续)

- 你能简述bin-packing problem的含义吗？

Input: n rational numbers $w_1, w_2, \ldots, w_n \in [0,1]$ for some positive integer n.

Constraints: $\mathcal{M}(w_1, w_2, \ldots, w_n) = \{S \subseteq \{0,1\}^n \mid$ for every $s \in S$, $s^{\mathsf{T}} \cdot (w_1, w_2, \ldots, w_n) \leq 1$, and $\sum_{s \in S} s = (1, 1, \ldots, 1)\}$.
{If $S = \{s_1, s_2, \ldots, s_m\}$, then $s_i = (s_{i1}, s_{i2}, \ldots, s_{in})$ determines the set of objects packed in the ith bin. The jth object is packed into the ith bin if and only if $s_{ij} = 1$. The constraint

$$s_i^{\mathsf{T}} \cdot (w_1, \ldots, w_n) \leq 1$$

assures that the ith bin is not overfilled. The constraint

$$\sum_{s \in S} s = (1, 1, \ldots, 1)$$

assures that every object is packed in exactly one bin.}

Cost: For every $S \in \mathcal{M}(w_1, w_2, \ldots, w_n)$,

$$cost(S, (w_1, \ldots, w_n)) = |S|.$$

Goal: *minimum*.

问题2：判定和优化问题 (续)

- 你能简述makespan scheduling problem的含义吗？

Input: Positive integers $p_1, p_2, \ldots, p_n$ and an integer $m \geq 2$ for some $n \in \mathbb{N} - \{0\}$.
$\{p_i$ is the processing time of the ith job on any of the m available machines$\}$.

Constraints: For every input instance $(p_1, \ldots, p_n, m)$ of MS,
$\mathcal{M}(p_1, \ldots, p_n, m) = \{S_1, S_2, \ldots, S_m \mid S_i \subseteq \{1, 2, \ldots, n\}$ for $i = 1, \ldots, m$, $\bigcup_{k=1}^{m} S_k = \{1, 2, \ldots, n\}$, and $S_i \cap S_j = \emptyset$ for $i \neq j\}$.
$\{\mathcal{M}(p_1, \ldots, p_n, m)$ contains all partitions of $\{1, 2, \ldots, n\}$ into m subsets. The meaning of $(S_1, S_2, \ldots, S_m)$ is that, for $i = 1, \ldots, m$, the jobs with indices from S_i have to be processed on the ith machine$\}$.

Costs: For each $(S_1, S_2, \ldots, S_m) \in \mathcal{M}(p_1, \ldots, p_n, m)$,
$cost((S_1, \ldots, S_m), (p_1, \ldots, p_n, m)) = \max\left\{\sum_{l \in S_i} p_l \mid i = 1, \ldots, m\right\}$.

Goal: *minimum*.

问题2：判定和优化问题 (续)

- 你能简述set cover problem的含义吗？

Input: $(X, \mathcal{F})$, where X is a finite set and $\mathcal{F} \subseteq Pot(X)$ such that $X = \bigcup_{S \in \mathcal{F}} S$.

Constraints: For every input $(X, \mathcal{F})$, $\mathcal{M}(X, \mathcal{F}) = \{C \subseteq \mathcal{F} \mid X = \bigcup_{S \in C} S\}$.

Costs: For every $C \in \mathcal{M}(X, \mathcal{F})$, $cost(C, (X, \mathcal{F})) = |C|$.

Goal: $minimum$.

问题2：判定和优化问题 (续)

- 你能简述maximum satisfiability problem的含义吗？

Input: A formula $\Phi = F_1 \wedge F_2 \wedge \cdots \wedge F_m$ over $X = \{x_1, x_2, \ldots\}$ in CNF (an equivalent description of this instance of MAX-SAT is to consider the set of clauses $F_1, F_2, \ldots, F_m$).

Constraints: For every formula Φ over the set $\{x_1, \ldots, x_n\} \subseteq X$, $n \in \mathbb{N} - \{0\}$, $\mathcal{M}(\Phi) = \{0,1\}^n$.
{Every assignment of values to $\{x_1, \ldots, x_n\}$ is a feasible solution, i.e., $\mathcal{M}(\Phi)$ can also be written as $\{\alpha \mid \alpha : X \to \{0,1\}\}$}.

Costs: For every Φ in CNF, and every $\alpha \in \mathcal{M}(\Phi)$, $cost(\alpha, \Phi)$ is the number of clauses satisfied by α.

Goal: *maximum*.

问题2：判定和优化问题 (续)

- 你能简述integer linear programming的含义吗？

Input: An $m \times n$ matrix $A = [a_{ij}]_{i=1,\ldots,m, j=1,\ldots,n}$, and two vectors $b = (b_1, \ldots, b_m)^\mathsf{T}$, $c = (c_1, \ldots, c_n)^\mathsf{T}$ for some $n, m \in \mathbb{N} - \{0\}$, a_{ij}, b_i, c_j are integers for $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Constraints: $\mathcal{M}(A, b, c) = \{X = (x_1, \ldots, x_n) \in \mathbb{Z}^n \mid AX = b \text{ and } x_i \geq 0$ for $i = 1, \ldots, n\}$.

Costs: For every $X = (x_1, \ldots, x_n) \in \mathcal{M}(A, b, c)$, $cost(X, (A, b, c)) = \sum_{i=1}^{n} c_i x_i$.

Goal: *minimum*.

问题2：判定和优化问题 (续)

- 你能简述maximum linear equation problem mod *k*的含义吗？

Input: A set S of m linear equations over n unknowns, $n, m \in \mathbb{N} - \{0\}$,
with coefficients from $\mathbb{Z}_k$.
(An alternative description of an input is an $m \times n$ matrix over $\mathbb{Z}_k$
and a vector $b \in \mathbb{Z}_k^m$).

Constraints: $\mathcal{M}(S) = \mathbb{Z}_k^m$
{a feasible solution is any assignment of values from $\{0, 1, \ldots, k-1\}$
to the n unknowns (variables)}.

Costs: For every $X \in \mathcal{M}(S)$,
$cost(X, S)$ is the number of linear equations of S satisfied by X.

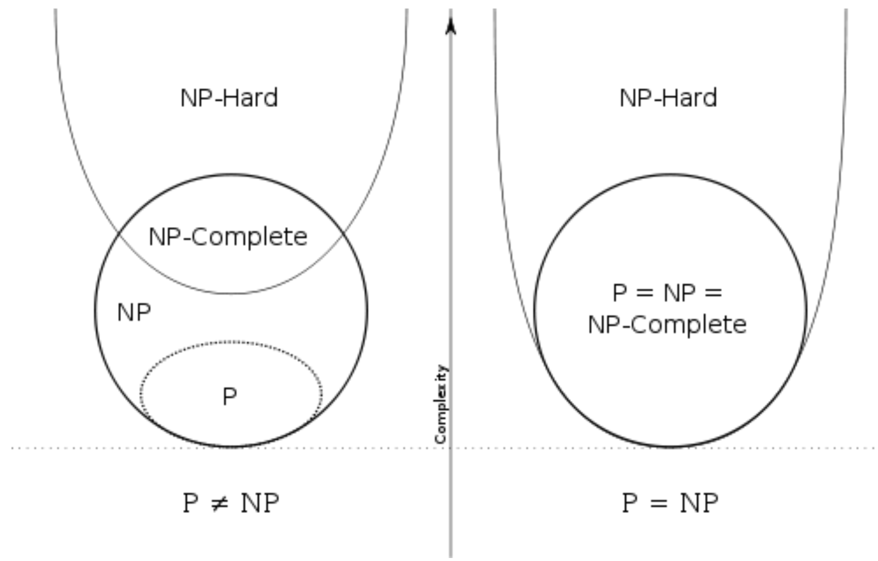
Goal: *maximum*.

问题3：P和NP

- 你怎么理解upper bound和lower bound？

问题3：P和NP (续)

- 你能解释清楚这些概念及其之间的关系吗？
  - P
  - NP
  - NP-hard
  - NP-complete



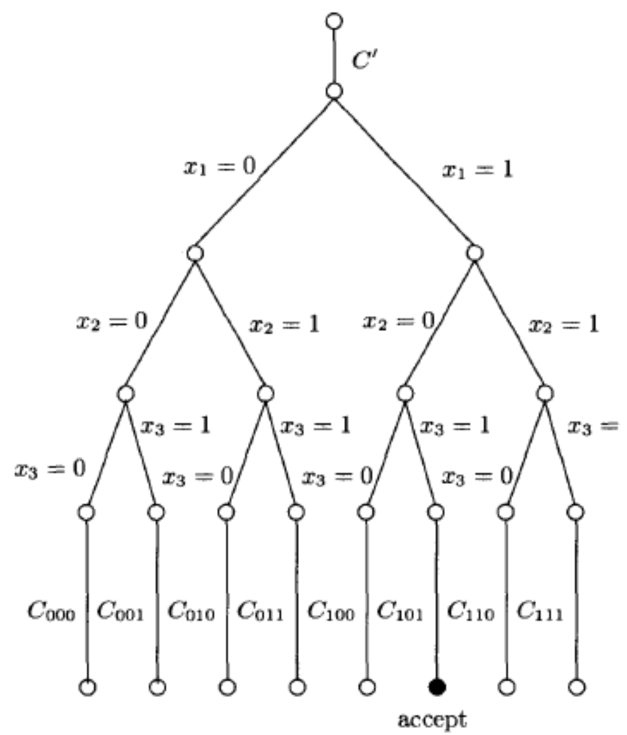
- 注意，经常被忽视的一点是，严格来说
  - P、NP、NP-complete都是描述判定问题的
  - NP-hard可以描述判定问题、优化问题等各类问题

问题3：P和NP (续)

- 你怎么理解这段话？

The complexity of deterministic computations is the complexity of proving the correctness of the produced output, while the complexity of nondeterministic computation is equivalent to the complexity of deterministic verification of a given proof (certificate) of the fact $x \in L$.

- 你能结合这个例子来解释吗？



问题3：P和NP (续)

- 优化问题也有自己的“NP”和“P”，你理解了吗？

Definition 2.3.3.21. NPO *is the class of optimization problems, where* $U = (\Sigma_I, \Sigma_O, L, L_I, \mathcal{M}, cost, goal) \in \text{NPO}$ *if the following conditions hold:*

(i) $L_I \in \text{P}$,
(ii) there exists a polynomial p_U *such that*
   a) for every $x \in L_I$, *and every* $y \in \mathcal{M}(x)$, $|y| \leq p_U(|x|)$, *and*
   b) there exists a polynomial-time algorithm that, for every $y \in \Sigma_O^*$ *and every* $x \in L_I$ *such that* $|y| \leq p_U(|x|)$, *decides whether* $y \in \mathcal{M}(x)$, *and*
(iii) the function cost is computable in polynomial time.

Informally, we see that an optimization problem U is in NPO if

(i) one can efficiently verify whether a string is an instance of U,
(ii) the size of the solutions is polynomial in the size of the problem instances and one can verify in polynomial time whether a string y is a solution to any given input instance x, and
(iii) the cost of any solution can be efficiently determined.

Definition 2.3.3.23. PO *is the class of optimization problems* $U = (\Sigma_I, \Sigma_O, L, L_I, \mathcal{M}, cost, goal)$ *such that*

(i) $U \in \text{NPO}$, *and*
(ii) there is a polynomial-time algorithm that, for every $x \in L_I$, *computes an optimal solution for* x.