Counterfeit Coin Problem (n coins) ①

⓪ "lighter" version (2019.10.10 周2). 1

① "labelled" version (P.H, P.L)

$n = a + b$.

② "unknown" version

③ "+standard coin" version.

A single weighing of any sort cannot do better than cut the size of the set of "suspect" coins by a factor of 3.

重点：在解决②时，各version之间是如何转换的

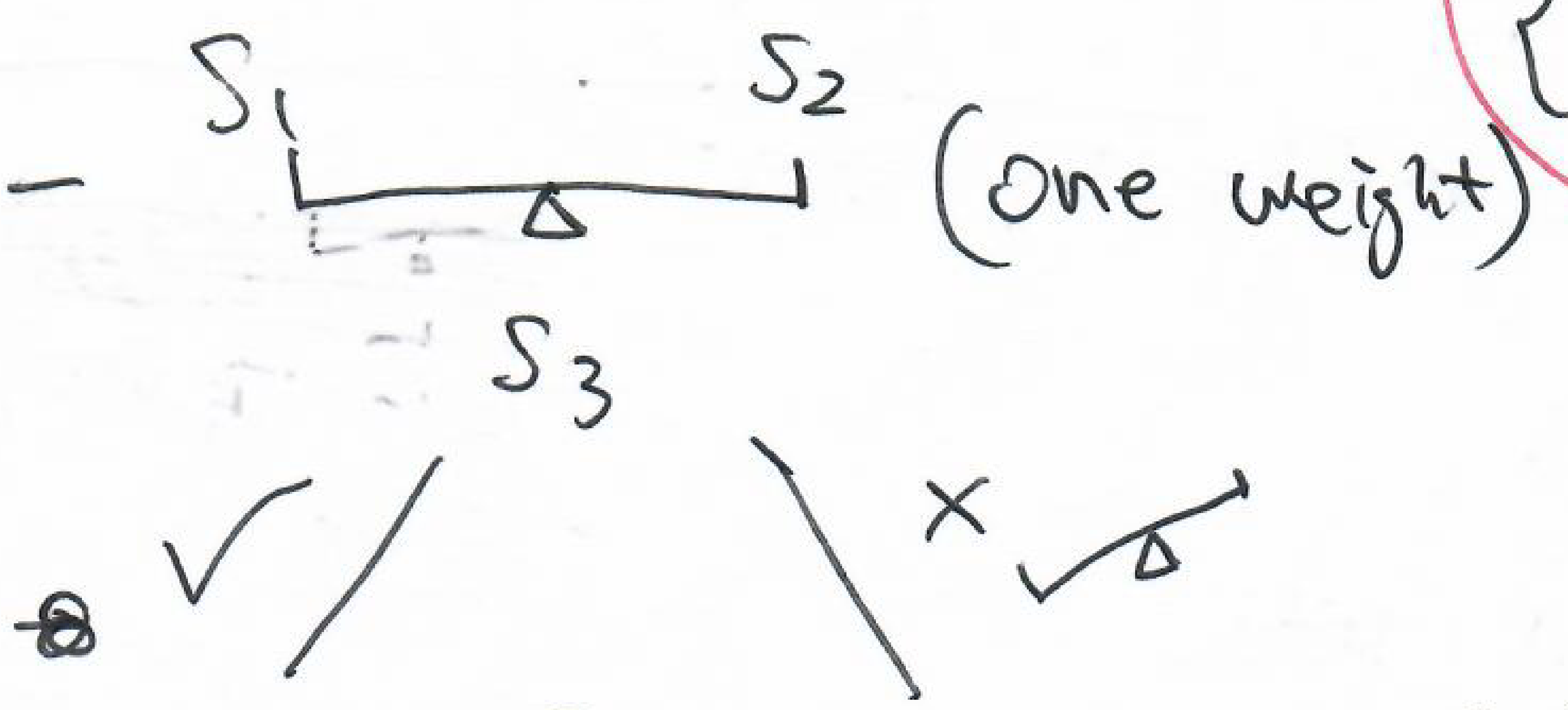
先看 "lighter" version.

n.

Algorithm (算法). (approximately).

- Divide ~~split~~ n coins into three almost equal sets, S_1, S_2, S_3.

$$\begin{cases} n = 3k & k \quad k \quad k \\ n = 3k+1 & k \quad k+1 \quad k \\ n = 3k+2 & k+1 \quad k \quad k+1 \end{cases}$$

- 

recurse on S_3. recurse on S_1 (or S_2)

$\frac{n}{3}$ $\lceil \frac{n}{3} \rceil$ $T(1) = 0$, $T(2) = 1$

$$\boxed{T(n) \le 1 + T(\lceil \tfrac{n}{3} \rceil)}$$

$T(2) \le 1 + T(1) = 1$

$T(n) \le 1 + (1 + T\lceil \frac{\lceil \frac{n}{3} \rceil}{3} \rceil)$

$3^{m-1} < n \le 3^m$

$$\boxed{\begin{aligned} & m-1 < \log_3 n \le m \\ \Leftrightarrow\ & m = \lceil \log_3 n \rceil \end{aligned}}$$

$m = T(n) \le \lceil \log_3 n \rceil$

离散值? 连续值?

~~$\boxed{T(n) \ge 1 + T(\frac{n}{3})}$~~

~~$\Rightarrow T(n) \ge \lceil \log_3 n \rceil$~~

lowerbound.

"the least number of weighings".

Q: least: 1?

the worst case!

State space:

n

~~cut the~~

$(\frac{1}{3})^m \cdot n \le 1$

$m \ge \lceil \log_3 n \rceil$.

求下界

see: ProblemOverflow

Q: 如何来精确求解?

$T(n) \le \lceil \log_3 n \rceil$

Q: 数学归纳法验证

Q: standard coin 没有用处?

Summary: ~~如下~~ 算法 upper bound $T(n) \le ?\ \alpha$

下界 lower bound: $L(n) \ge ?\ \beta$

α β

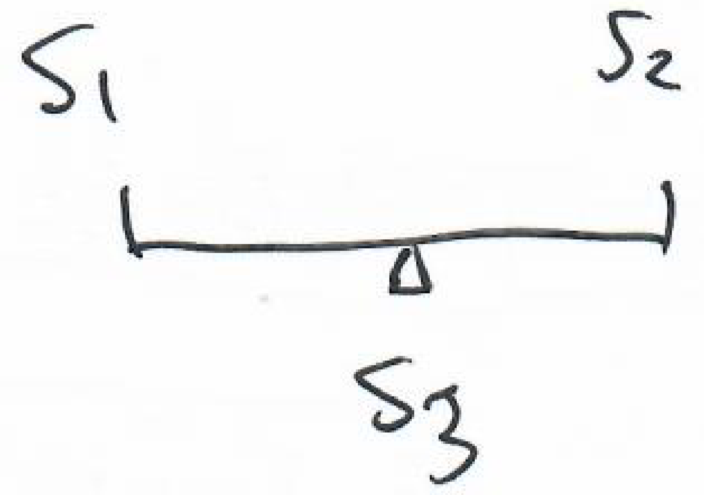
$\alpha = \beta$ 同时最优.

" labelled " version $\quad n = \underset{(P.H.)}{a} + \underset{(P.L.)}{b}$

②

$$m = \lceil \log_3 n \rceil .$$

Algorithm



x p.H. y p.L. x p.H y p.L.

(a−2x) pH. (b−2y) p.L.

" Whenever we place coins on the scale, we must be sure to put equal number of p.L. coins on the two sides and equal numbers of p.h. coins on the two sides.

a: three approximately equal sets

b: three approximately equal sets.

$$a = \begin{cases} 3k \\ 3k+1 \\ 3k+2 \end{cases} \qquad b = \begin{cases} 3k' \\ 3k'+1 \\ 3k'+2 \end{cases}$$

共9种情况. $3^2 = 9$.

由对称性 共6种情况 $C_{3+3}^{2} = 6$.

反例需要特别注意.

Example $a = 3k+1$ $b = 3k'+1$

$n = 3(k+k')+2$

$\boxed{\lceil \frac{n}{3} \rceil = k+k'+1}$

~~[illegible]~~
错误方法: k ; k' $\quad$ k ; k'

k+1, k'+1 $\quad$ X

$\boxed{k+k'+2}$

正确方法: k ; k'+1 $\quad$ k ; k'+1

(k+k'+1).

k+1 $\quad$ k'−1

k+k'

We want

$$T(n) \leq 1 + T(\lceil \tfrac{n}{3} \rceil)\ ?$$

$$\leq \lceil \log_3 n \rceil .$$

Lower bound = upper bound.

Q: 有点难理解... 验证.

Lower bound

Same as the " lighter " version

" Lighter " version is a special case of " labelled " every coin is labelled " p.L. ".

$$m \geq \lceil \log_3 n \rceil .$$

注: n=1; n=2 时 1 PH. 1 PL 无解

若递归到此情形, 可以"借"一枚 labelled coin, 变成 n=3 的情况.

"unknown" type. (n). 无论这个.

③

$n = 12$. ($m=4$: easy ; $m=3$: hard).

如果变成"light" version 需要 $m=4$.

4 4

4

第一次称量. 此时如果变成 the "light" version 需要 $m=4$.

We want $m \le 3$!

$m=3$

4 p.L ; 4 p.H. ; 4 standard coin.

"labelled" version for $a=4$, $b=4$

$\lceil \log_3 4 \rceil = 2$.

4 4

4.

recurses on 4 coins ???

$n=4$ "unknown" type.

$T(4) \le 2$ ⇐ We want this

$T(1)$ × no definitions

$T(2)$ ×

We require $n \ge 3$.

$n=4$. 只有2次 weighing 机会

2 2 ; 0 ; × × labelled version ② $\lceil \log_3 4 \rceil = 2$

1 1 ; ② ; × ✓ $\lceil \log_3 2 \rceil = 1$; $T(2)$ ×

只考虑实际的2个肯定不行

4 unknown coins ; "1" standard coin

第一次称量.

2 ; 1 ; 1 standard coin version.

1

× 2 pH 1 pL ; 2 pL. 1 pH

$\lceil \log_3 3 \rceil = 1$.

"labelled version"

✓ 2 ; 1 ; 1 ; 1

假币 真币

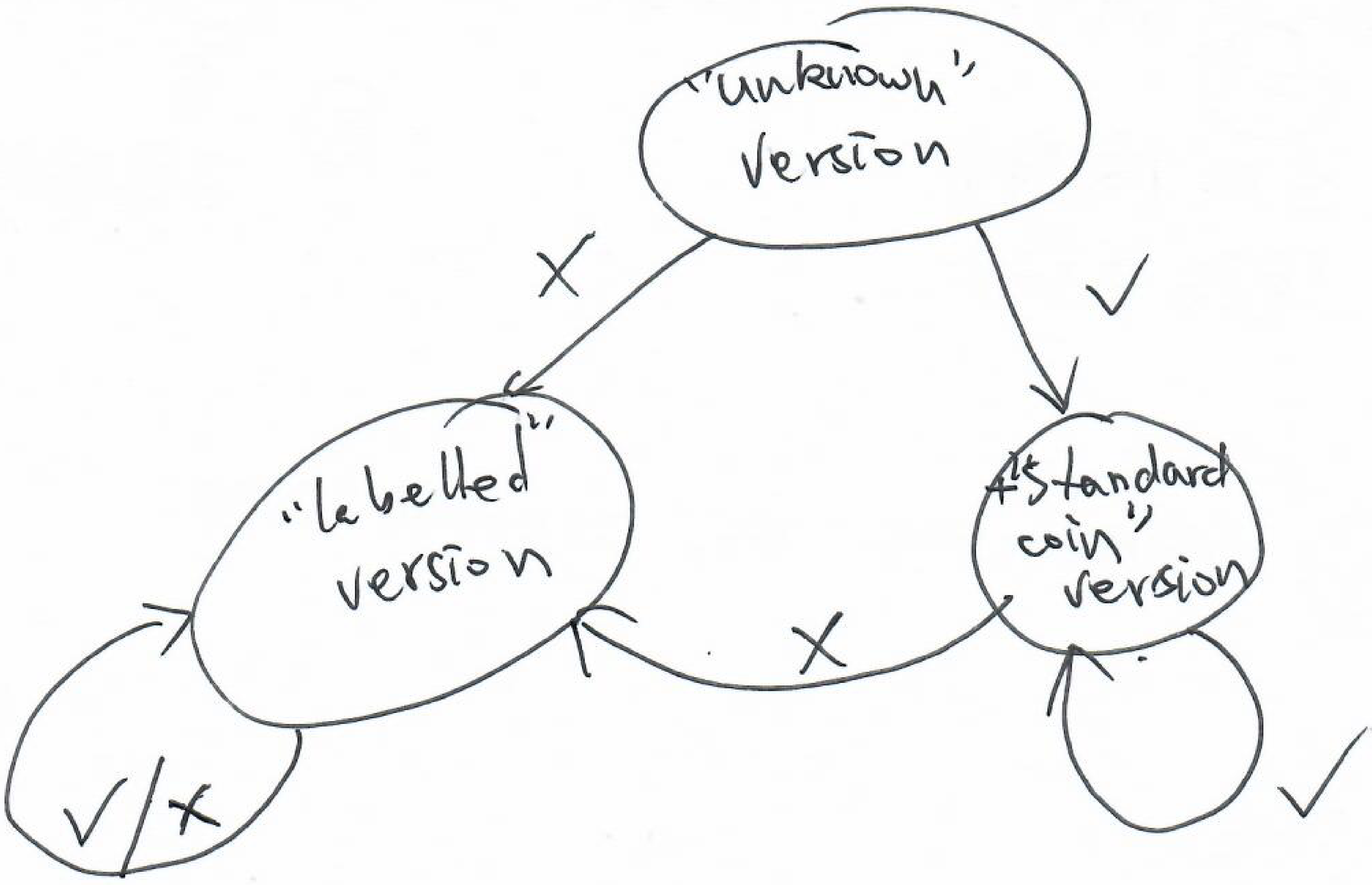
1 假币

主要是在确定轻重 standard coin 假币的重量.

1 真 1 unknown ⇒ 确认假币, 但无法确认轻重.

2 真 2 unknown. ⇒ 可确认 pH. pL 但无法确认假币.

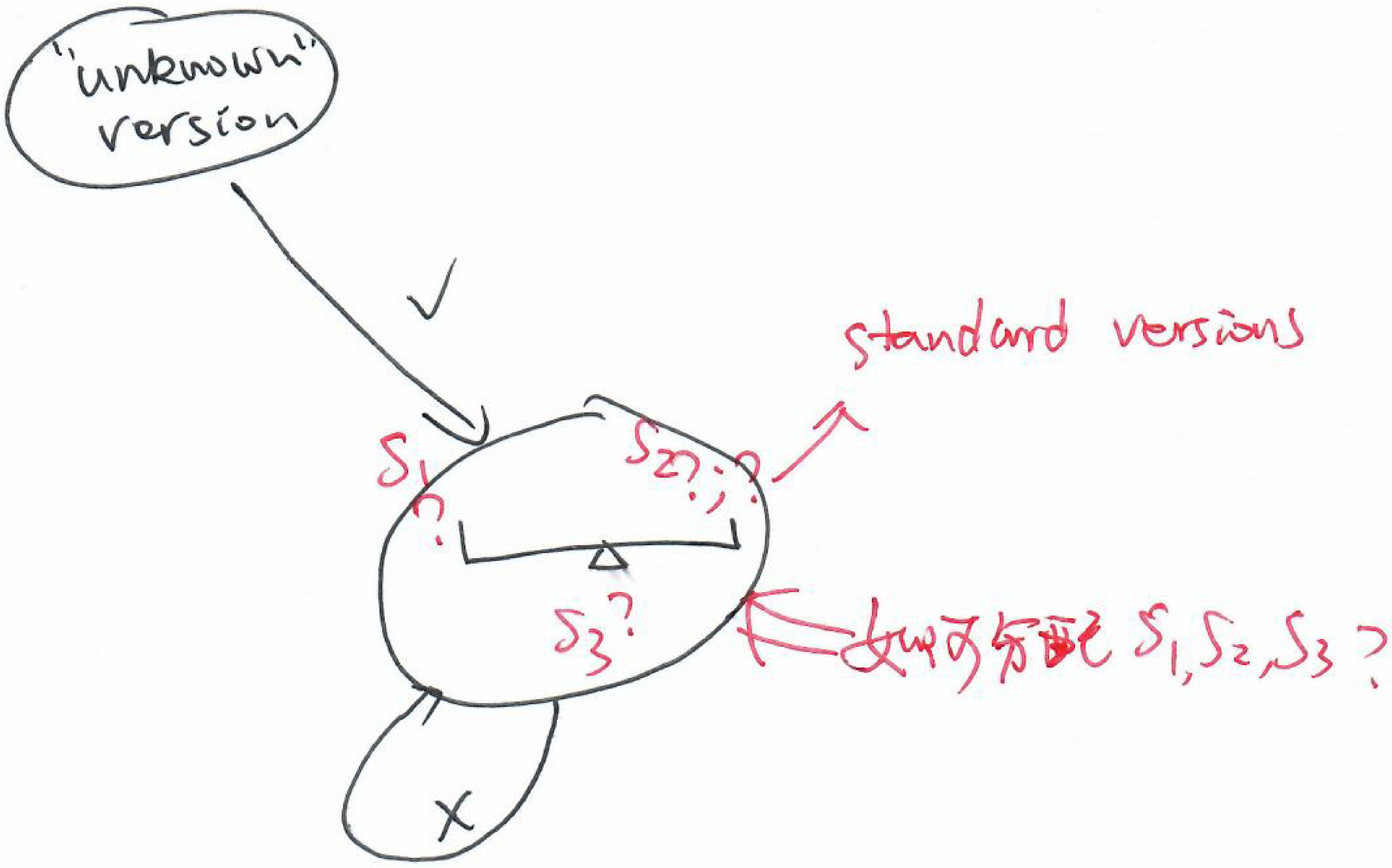
Summary



$$m: \quad (3^{m-1}-1)/2 < n \le (3^m-1)/2$$

$$\Rightarrow \quad m = \lceil \log_3(2n+1) \rceil .$$

关鍵設:



一旦"不平"，即进入"labelled" version，
該 version 易解。
否則一直处于" + standard version"中。