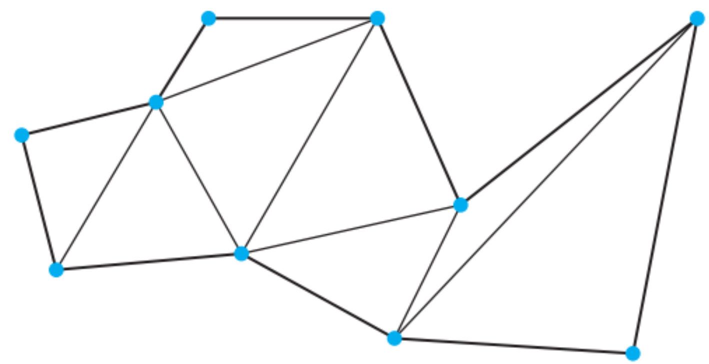
习题2-6

CS 4.1-16

4.1-16 An alternate version of the Ear Lemma states that a triangulated polygon is either a triangle with three ears or has at least two ears. (This version does not specify that the ears are nonadjacent.) What happens if we try proving this by induction, using the same decomposition that we used in proving the Ear Lemma?

- 如果要证明，如何证？
  - 数学归纳法+反证



习题2-7

CS 5.2-10,14,15
CS 5.4-8,21

5.2-10. If you are hashing n keys into a hash table with k locations, what is the probability that every location gets at least one key?

?

(1)n<k 时，概率为 0

(2)n>=k 时，可知总情况的数目有 k^n 个，即每个位置，总共都有 n 种 key 的可能，如果每个位置至少有 1 把 key，我们先从 n 个 key 中取 k 个，即 C(n,k)种取法，进行排列种类有 k!种，而剩余的(n-k)把 key，对于 k 个位置，一共有 k^(n-k)种可能性。因此，设每个位置至少有一把 key 为事件 A，则 $P(A)=\dfrac{C_n^k k!\cdot k^{n-k}}{k^n}=\dfrac{n!}{(n-k)!k^k}$

E_i: the i-th location has no key

$$N(\overline{E_1}\,\overline{E_2}\ldots\overline{E_k}) = |S| - \sum_{i=1}^{k}(-1)^{i+1}S_i$$

$$= |S| - \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\cdot(k-i)^n$$

$$= \sum_{i=0}^{k}(-1)^{i}\binom{k}{i}\cdot(k-i)^n$$

$$P(\overline{E_1}\,\overline{E_2}\ldots\overline{E_k}) = \frac{N(\overline{E_1}\,\overline{E_2}\ldots\overline{E_k})}{|S|}$$

$$|S| = |S_0| = k^n$$

$$S_1 = \binom{k}{1}\cdot(k-1)^n$$

$$S_1 = \binom{k}{2}\cdot(k-2)^n$$

…

$$S_i = \binom{k}{i}\cdot(k-i)^n$$

…

5.2-**14.** A group of n married couples sits around a circular table for a discussion of marital problems. The counselor assigns each person to a seat at random. What is the probability that no husband and wife are side-by-side?

- $|S| = (2n-1)!$
- E_i:第i对夫妇坐在一起
- 求: $N(\overline{E_1}\,\overline{E_2}\,\ldots\overline{E_n}) = |S| - \sum_{k=1}^{n}(-1)^{k+1}S_k$
- S_k: 至少有k对夫妇坐在一起的安排集合（有重复）
  - $S_k = ?$
  - $S_k = \binom{n}{k}\cdot 2^k\cdot(2n-k-1)$
  - $S_0 = |S|$
- $N(\overline{E_1}\,\overline{E_2}\,\ldots\overline{E_n}) = |S| - \sum_{k=1}^{n}(-1)^{k+1}S_k = \sum_{k=0}^{n}(-1)^{k}S_k = \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\cdot 2^k\cdot(2n-k-1)$

5.2- 15. Suppose you have a collection of m objects and a set P of p "properties." (We won't define the term "property," but note that a property is something the objects may or may not have.) For each subset S of the set P of all properties, define $N_a(S)$ to be the number of objects in the collection that have *at least* the properties in S (a is for "at least"). Thus, for example, $N_a(\emptyset) = m$. In a typical application, formulas for $N_a(S)$ for other sets $S \subseteq P$ are not difficult to figure out. Define $N_e(S)$ to be the number of objects in our collection that have *exactly* the properties in S (e is for "exactly"). Show that

$$N_e(\emptyset) = \sum_{K:K \subseteq P} (-1)^{|K|} N_a(K).$$

Explain how this formula could be used to compute the number of onto functions in a more direct way than we did when using unions of sets. How would this formula apply to Problem 9?

$N_e(\emptyset)$: the number of objects in the collection having non property in P

$$N_e(\emptyset) = |S| - \sum_{k=1}^{n} (-1)^{k+1} S_k$$

其中$|S| = m$, $S_k = \sum_{1 \le i_1 \le i_2 \le \cdots i_k \le n,} |E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}|$, k=1,2,3…,n
E_i: the element in S has property $p_i \in P$

又∵ $\left|E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}\right| = N_a\left(\{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}\right)$
$\therefore S_k = \sum_{K_k \subseteq P, |K_k| = k} N_a(K_k)$
$\therefore N_e(\emptyset) = |S| - \sum_{k=1}^{n} (-1)^{k+1} \left[\sum_{K_k \subseteq P, |K_k| = k} N_a(K_k)\right]$
$= |S| - \sum_{k=1}^{n} \left[\sum_{K_k \subseteq P, |K_k| = k} (-1)^{k+1} N_a(K_k)\right]$
$= |S| - \sum_{k=1}^{n} \left[\sum_{K_k \subseteq P, |K_k| = k} (-1)^{|K_k|+1} N_a(K_k)\right]$
$= |S| - \sum_{K \subseteq P, |K| > 0} (-1)^{|K|+1} N_a(K)$
$= \sum_{K \subseteq P} (-1)^{|K|} N_a(K)$

5.4- 8. If you randomly choose 26 cards from a deck of 52 ordinary playing cards, one at a time, is the event of having a king on the ith draw independent of the event of having a king on the jth draw? How many kings do you expect to see?

$$|S| = 52 \cdot 51 \cdot \cdots \cdot 27 = \frac{52!}{26!}$$

E_x = having a king on the x th draw

$$|E_x| = \binom{4}{1} \cdot \frac{51!}{26!}, P(E_x) = \frac{|E_x|}{|S|} = \frac{4}{52} = \frac{1}{13}$$

$$P(E_i) = P(E_j) = P(E_x)$$

$$|E_i \cap E_j| = \binom{4}{2} \cdot 2! \cdot \frac{50!}{26!}, P(E_i \cap E_j) = \frac{|E_i \cap E_j|}{|S|} = \frac{3}{13 \cdot 51}$$

而$P(E_i)P(E_j) = \frac{1}{13} \cdot \frac{1}{13}$

$$P(E_i)P(E_j) \neq P(E_i \cap E_j)$$

5.4- 21. Give an example of a random variable on the sample space $\{S, FS, FFS, \ldots, F^i S, \ldots\}$ with an infinite expected value, using a geometric distribution for probabilities of $F^i S$.

设$A = \{S, FS, FFS, \ldots, F^i S, \ldots\}$

$X: A \to R$

$$E(X) = \sum_{e \in A} X(e) \cdot P(e) = \sum_{i=0}^{\infty} X(F^i S) \cdot P(F^i S)$$

$$= \sum_{i=0}^{\infty} X(F^i S) \cdot (1-p)^i p$$

$$= p \sum_{i=0}^{\infty} X(F^i S) \cdot (1-p)^i$$

如果$X(F^i S) \cdot (1-p)^i \geq \frac{1}{i}$,则$E(X)$不收敛，即$X(F^i S) \geq \frac{1}{i(1-p)^i}$即可