1-9 Set Theory (II): Relations

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Set Theory

A Branch of Mathematics

N, R

\omega

\mathbb{N}_0

Foundation of Mathematics

(+ Logic)

(a, b)

\{\}

f : A \rightarrow B

A \times B

R \subseteq A \times B

Hengfeng Wei (hfwei@nju.edu.cn)

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Figure 13. A selection of consistency axioms over an execution \((E, \text{repl}, \text{obj}, \text{oper}, \text{rval}, \text{ro}, \text{vis}, \text{ar})\).

Auxiliary relations

\[\text{sameobj}(e, f) \iff \text{obj}(e) = \text{obj}(f)\]

Per-object causality (aka happens-before) order:

\[\text{hbo} = ((\text{ro} \cap \text{sameobj}) \cap \text{vis})^+\]

Causality (aka happens-before) order: \(\text{hb} = (\text{ro} \cup \text{vis})^+\)

Axioms

EVENTUAL:

\[\forall e \in E. \neg(\exists \text{ infinitely many } f \in E. \text{sameobj}(e, f) \land \neg(e \rightsquigarrow f))\]

THINAIR: \(\text{ro} \cup \text{vis} \text{ is acyclic}\)

POCV (Per-Object Causal Visibility): \(\text{hbo} \subseteq \text{vis}\)

POCA (Per-Object Causal Arbitration): \(\text{hbo} \subseteq \text{ar}\)

COCV (Cross-Object Causal Visibility): \((\text{hb} \cap \text{sameobj}) \subseteq \text{vis}\)

COCA (Cross-Object Causal Arbitration): \(\text{hb} \cup \text{ar} \text{ is acyclic}\)

Assume \((v, w) \in [E, V, \text{rval}, \text{ro}, \text{vis}, \text{ar})\).

\[\text{sameobj}(e, f) \iff \text{obj}(e) = \text{obj}(f)\]

By agree we have \(\text{ro} \cup \text{vis} \subseteq \text{Ex}.\) Then

\[\{\forall (a, v', (a', v') \in V. (a = a' \iff v = v') \land \neg(\exists \text{ infinitely many } f \in E. \text{sameobj}(a, f) \land \neg(a \rightsquigarrow f))\}, \text{Ex}\}

\[\{\forall (a, v', (a', v') \in V. (a = a' \iff v = v') \land \neg(\exists \text{ infinitely many } f \in E. \text{sameobj}(a, f) \land \neg(a \rightsquigarrow f))\}, \text{Ex}\}

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Figure 13. A selection of consistency axioms over an execution $(E, \text{repl. obj, oper, rval, ro, vis, ar})$

Auxiliary relations

- **sameobj($e, f$) $\iff$ obj($e$) = obj($f$)**
- **Per-object causality (aka happens-before) order:**
  
  $\text{hbo} = (\text{ro} \cap \text{sameobj}) \cup \text{vis}^{+}$

- **Causality (aka happens-before) order:**
  
  $\text{hb} = (\text{ro} \cup \text{vis})^{+}$

**Axioms**

**Eventual:**

$\forall e \in E. \neg(\exists \text{ infinitely many } f \in E. \text{sameobj}(e, f) \land \neg(e \mathbin{\xrightarrow{\text{vis}}} f))$

**Thinair:**

$\text{ro} \cup \text{vis}$ is acyclic

**POCV (Per-Object Causal Visibility):**

$hbo \subseteq \text{vis}$

**POCA (Per-Object Causal Arbitration):**

$hbo \subseteq \text{ar}$

**COCV (Cross-Object Causal Visibility):**

$(hbo \cap \text{sameobj}) \subseteq \text{vis}$

**COCA (Cross-Object Causal Arbitration):**

$hbo \cup \text{ar}$ is acyclic

Assume $(e, V) \in [E \times V] [\text{vis}]$, and

$f = (E', \text{repl. obj, oper, rval, ro, vis, ar})$.

thus

$\forall e \in E'. \text{sameobj}(e', f) \land \neg(e' \mathbin{\xrightarrow{\text{vis}}} f)$
DON'T BE SCARED

AN ANTHOLOGY FOR CHILDREN BY WELL-LOVED AUTHORS & ARTISTS
I’m so excited.
Definition (Relations)

A *relation* $R$ from $A$ to $B$ is a subset of $A \times B$:

$$R \subseteq A \times B$$
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Definition (Cartesian Products)

The *Cartesian product* $A \times B$ of $A$ and $B$ is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \land b \in B\}$$
**Definition (Relations)**

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**Axiom (Ordered Pairs)**

\( (a, b) = (c, d) \iff a = c \land b = d \)
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A relation $R$ from $A$ to $B$ is a subset of $A \times B$:

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Axiom (Ordered Pairs)

$$(a, b) = (c, d) \iff a = c \land b = d$$

Q: Are you satisfied with the definitions above?
Axiom (Ordered Pairs)

\((a, b) = (c, d) \iff a = c \land b = d\)
Axiom (Ordered Pairs)

\[(a, b) = (c, d) \iff a = c \land b = d\]

Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

\[(a, b) \triangleq \{\{a\}, \{a, b\}\}\]
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\[(a, b) \triangleq \{\{a\}, \{a, b\}\}\]

Theorem

\[(a, b) = (c, d) \iff a = c \wedge b = d\]
Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

\[(a, b) \triangleq \{\{a\}, \{a, b\}\}\]

Theorem

\[(a, b) = (c, d) \iff a = c \land b = d\]

Proof.

\[\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}\]
Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

\[(a, b) \triangleq \{\{a\}, \{a, b\}\}\]

Theorem

\[(a, b) = (c, d) \iff a = c \land b = d\]

Proof.

\[
\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}
\]

\textbf{CASE I} : \(a = b\)

\textbf{CASE II} : \(a \neq b\)
Definition (Ordered Pairs (Norbert Wiener; 1914))

\[(a, b) \triangleq \{\{a\}, \emptyset, \{\{b\}\}\}\]
Definition (Ordered Pairs (Norbert Wiener; 1914))

\[(a, b) \triangleq \left\{ \{\{a\}, \emptyset\}, \{\{b\}\} \right\}\]

Theorem

\[(a, b) = (c, d) \iff a = c \land b = d\]
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The *Cartesian product* $A \times B$ of $A$ and $B$ is defined as

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Theorem

$A \times B$ is a set.
Definition (Cartesian Products)

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$$X^2 \triangleq X \times X$$

Theorem

$A \times B$ is a set.

Proof.

$$A \times B \triangleq \{ (a, b) \in \mathcal{P} \mid a \in A \land b \in B \}$$
Definition (Cartesian Products)

The *Cartesian product* $A \times B$ of $A$ and $B$ is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \land b \in B\}$$

$$X^2 \triangleq X \times X$$

Theorem

$A \times B$ is a set.

Proof.

$$A \times B \triangleq \{(a, b) \in ? \mid a \in A \land b \in B\}$$

$$\{\{a\}, \{a, b\}\} \in ?$$
Definition (Cartesian Products)

The *Cartesian product* $A \times B$ of $A$ and $B$ is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \land b \in B\}$$

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Theorem

$A \times B$ is a set.

Proof.

$$A \times B \triangleq \{(a, b) \in \? \mid a \in A \land b \in B\}$$

$$\\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$$
Definition (Relations)

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If $A = B$, $R$ is called a relation on $A$. 
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If $A = B$, $R$ is called a relation on $A$.

Definition (Notations)

$$(a, b) \in R \quad R(a, b) \quad aRb$$
Definition (Relations)

A relation $R$ from $A$ to $B$ is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples
Definition (Relations)

A *relation* $R$ from $A$ to $B$ is a subset of $A \times B$:

$$R \subseteq A \times B$$

**Examples**

- Both $A \times B$ and $\emptyset$ are relations from $A$ to $B$. 
Definition (Relations)

A relation $R$ from $A$ to $B$ is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples

- Both $A \times B$ and $\emptyset$ are relations from $A$ to $B$.

- $< = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b\}$
Definition (Relations)

A \textit{relation} $R$ from $A$ to $B$ is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples

\begin{itemize}
  \item Both $A \times B$ and $\emptyset$ are relations from $A$ to $B$.
  \item $< = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b\}$
  \item $D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N} : a \cdot q = b\}$
\end{itemize}
Definition (Relations)

A relation $R$ from $A$ to $B$ is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples

▸ Both $A \times B$ and $\emptyset$ are relations from $A$ to $B$.

▸

$$< = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b\}$$

▸

$$D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N} : a \cdot q = b\}$$

▸ $P$ : the set of people

$$M = \{(a, b) \in P \times P \mid a \text{ is the mother of } b\}$$

$$B = \{(a, b) \in P \times P \mid a \text{ is the brother of } b\}$$
Important Relations:

Equivalence Relations (1-9)

Functions (1-10)

Ordering Relations (1-12)
Before that,

3 Definitions

5 Operations

7 Properties

\[ R = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 3)\} \]
3 Definitions
Definition (Domain)

\[ \text{dom}(R) = \{ a \mid \exists b : (a, b) \in R \} \]
Definition (Domain)

\[ \text{dom}(R) = \{ a \mid \exists b : (a, b) \in R \} \]

Theorem

\( \text{dom}(R) \) is a set.
Definition (Domain)

\[ \text{dom}(R) = \{ a \mid \exists b : (a, b) \in R \} \]

Theorem

\[ \text{dom}(R) \text{ is a set.} \]

\[ \text{dom}(R) = \{ a \in \text{?} \mid \exists b : (a, b) \in R \} \]
Definition (Domain)

\[ \text{dom}(R) = \{ a \mid \exists b : (a, b) \in R \} \]

Theorem

\text{dom}(R) \text{ is a set.}

\[ \text{dom}(R) = \{ a \in \ ? \mid \exists b : (a, b) \in R \} \]

\[ (a, b) = \{ \{a\}, \{a, b\} \} \in R \]
Definition (Domain)

\[ \text{dom}(R) = \{ a \mid \exists b : (a, b) \in R \} \]

Theorem

\( \text{dom}(R) \) is a set.

\[ \text{dom}(R) = \{ a \in ? \mid \exists b : (a, b) \in R \} \]

\[ (a, b) = \{ \{a\}, \{a, b\} \} \in R \]

\[ \{a, b\} \in \bigcup R \]
Definition (Domain)

\[ \text{dom}(R) = \{ a \mid \exists b : (a, b) \in R \} \]

Theorem

\[ \text{dom}(R) \text{ is a set.} \]

\[ \text{dom}(R) = \{ a \in ? \mid \exists b : (a, b) \in R \} \]

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Definition (Domain)

\[ \text{dom}(R) = \{ a \mid \exists b : (a, b) \in R \} \]

Theorem

\( \text{dom}(R) \) is a set.

\[ \text{dom}(R) = \{ a \in \bigcup \bigcup R \mid \exists b : (a, b) \in R \} \]

\((a, b) = \{\{a\}, \{a, b\}\} \in R\)

\(\{a, b\} \in \bigcup R\)

\(a \in \bigcup \bigcup R\)
Definition (Range)

\[ \text{ran}(R) = \{ b \mid \exists a : (a, b) \in R \} \]
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\[ \text{ran}(R) = \{ b \mid \exists a : (a, b) \in R \} \]

Theorem

\text{ran}(R) \text{ is a set.}

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Definition (Range)

\[ \text{ran}(R) = \{b \mid \exists a : (a, b) \in R\} \]

Theorem

\( \text{ran}(R) \) is a set.

\[ \text{ran}(R) = \{b \in \bigcup \bigcup R \mid \exists a : (a, b) \in R\} \]

Definition (Field)

\[ \text{fld}(R) = \text{dom}(R) \cup \text{ran}(R) \]
5 Operations
Definition (Inverse)
The *inverse* of $R$ is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$
Definition (Inverse)

The *inverse* of $R$ is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

Theorem

$$(R^{-1})^{-1} = R$$
Definition (Inverse)
The inverse of $R$ is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

Theorem

$$(R^{-1})^{-1} = R$$

Definition (Restriction)
The restriction of $R$ to $X$ is the relation

$$R|_X = \{(a, b) \in R \mid a \in X\}$$
Definition (Image)
The *image* of $X$ under $R$ is the set

$$R[X] = \{ b \in \text{ran}(R) \mid \exists a \in X : (a, b) \in R \}$$
Definition (Image)

The *image* of $X$ under $R$ is the set

$$R[X] = \{ b \in \text{ran}(R) \mid \exists a \in X : (a, b) \in R \} = \text{ran}(R|_X)$$
Definition (Image)
The image of $X$ under $R$ is the set

$$R[X] = \{ b \in \text{ran}(R) \mid \exists a \in X : (a, b) \in R \} = \text{ran}(R|_X)$$

Definition (Inverse Image)
The inverse image of $Y$ under $R$ is the set

$$R^{-1}[Y] = \{ b \in \text{dom}(R) \mid \exists b \in Y : (a, b) \in R \}$$
Definition (Image)

The *image* of $X$ under $R$ is the set

$$R[X] = \{ b \in \text{ran}(R) \mid \exists a \in X : (a, b) \in R \} = \text{ran}(R|_X)$$

Definition (Inverse Image)

The *inverse image* of $Y$ under $R$ is the set

$$R^{-1}[Y] = \{ b \in \text{dom}(R) \mid \exists b \in Y : (a, b) \in R \} = \text{ran}(R^{-1}|_Y)$$
$R \subseteq A \times B \quad X \subseteq A \quad Y \subseteq B$
$R \subseteq A \times B \quad X \subseteq A \quad Y \subseteq B$

$R^{-1}[R[X]] \subseteq X$

$R[R^{-1}[Y]] \subseteq Y$
$R \subseteq A \times B \quad X \subseteq A \quad Y \subseteq B$

$R^{-1}[R[X]] \neq X$

$R[R^{-1}[Y]] \neq Y$
Theorem

\[ R[X_1 \cup X_2] = R[X_1] \cup R[X_2] \]

\[ R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2] \]

\[ R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2] \]
Theorem

\[ R[X_1 \cup X_2] = R[X_1] \cup R[X_2] \]

\[ R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2] \]

\[ R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2] \]

\[ b \in R[X_1 \cup X_2] \]
Theorem

\[ R[X_1 \cup X_2] = R[X_1] \cup R[X_2] \]

\[ R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2] \]

\[ R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2] \]

\[ b \in R[X_1 \cup X_2] \iff \exists a \in X_1 \cup X_2 : (a, b) \in R \]
Theorem

\[ R[X_1 \cup X_2] = R[X_1] \cup R[X_2] \]

\[ R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2] \]

\[ R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2] \]

\[ b \in R[X_1 \cup X_2] \iff \exists a \in X_1 \cup X_2 : (a, b) \in R \]

\[ \iff \exists a \in X_1 : (a, b) \in R \lor \exists a \in X_2 : (a, b) \in R \]
Theorem

\[ R[X_1 \cup X_2] = R[X_1] \cup R[X_2] \]

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\[ b \in R[X_1 \cup X_2] \iff b \in R[X_1] \lor b \in R[X_2] \]
Definition (Composition)

The *composition* of relations \( R \) and \( S \) is the relation

\[
R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}
\]
Definition (Composition)

The \textit{composition} of relations $R$ and $S$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 3)\}$$
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The *composition* of relations \( R \) and \( S \) is the relation

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R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}
\]

\[
R = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 3)\}
\]

\[
R \circ R = \{\cdots\}
\]
Definition (Composition)

The composition of relations $R$ and $S$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 3)\}$$

$$R \circ R = \{\cdots\}$$

$$\leq \circ \leq =$$
Definition (Composition)

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$$R = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 3)\}$$

$$R \circ R = \{\cdots\}$$

$$\leq \circ \leq = \leq$$
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The \textit{composition} of relations $R$ and $S$ is the relation

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$$R = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 3)\}$$

$$R \circ R = \{\cdots\}$$

$$\leq \circ \leq = \leq$$

$$\leq \circ \geq =$$
Definition (Composition)

The *composition* of relations $R$ and $S$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 3)\}$$

$$R \circ R = \{\cdots\}$$

$$\leq \circ \leq = \leq$$

$$\leq \circ \geq = R \times R$$
Theorem

\[(R \circ S)^{-1} = S^{-1} \circ R^{-1}\]
Theorem

\[(R \circ S)^{-1} = S^{-1} \circ R^{-1}\]

\[(a, b) \in (R \circ S)^{-1} \iff \ldots\]
Theorem

\[(R \circ S) \circ T = R \circ (S \circ T)\]
Theorem

\[(R \circ S) \circ T = R \circ (S \circ T)\]

\[(a, b) \in (R \circ S) \circ T \iff \cdots\]
\[(a, b) \in (R \circ S) \circ T\]
\[(a, b) \in (R \circ S) \circ T\]
\[\iff \exists c : (a, c) \in T \land (c, b) \in R \circ S\]
\[(a, b) \in (R \circ S) \circ T \]
\[\iff \exists c : (a, c) \in T \land (c, b) \in R \circ S \]
\[\iff \exists c : (a, c) \in T \land (\exists d : (c, d) \in S \land (d, b) \in R) \]
\[(a, b) \in (R \circ S) \circ T\]
\[\iff \exists c : (a, c) \in T \land (c, b) \in R \circ S\]
\[\iff \exists c : (a, c) \in T \land (\exists d : (c, d) \in S \land (d, b) \in R)\]
\[\iff \exists d : \exists c : (a, c) \in T \land (c, d) \in S \land (d, b) \in R\]
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\[\iff \exists d : (\exists c : (a, c) \in T \land (c, d) \in S) \land (d, b) \in R\]
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\[\iff \exists d : (a, d) \in S \circ T \land (d, b) \in R\]
\[\iff (a, b) \in R \circ (S \circ T)\]
燕小六: “帮我照顾好我七舅姥爷和我外甥女”
“舅姥爷”：姥姥的兄弟
“舅姥爷”：姥姥的兄弟

\[ G = \{ (a, b) : a \text{ 是 } b \text{ 的舅姥爷} \} \]
“舅姥爷”：姥姥的兄弟

\[ G = \{ (a, b) : a \text{ 是 } b \text{ 的舅姥爷} \} \]

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\[ G = B \circ (M \circ M) \]
“舅姥爷”：姥姥的兄弟

$$G = \{(a, b) : a \text{ 是 b 的舅姥爷}\}$$

$$M = \{(a, b) \mid a \text{ is the mother of } b\}$$

$$B = \{(a, b) \mid a \text{ is the brother of } b\}$$

$$G = B \circ (M \circ M)$$

$$G = B \circ (M \circ M) = (B \circ M) \circ M$$
7 Properties
\( R \subseteq X \times X \)

**Definition (Reflexive)**

\[ \forall a \in X : (a, a) \in R \]
\[ R \subseteq X \times X \]

**Definition (Reflexive)**

\[ \forall a \in X : (a, a) \in R \]

**Definition (Irreflexive)**

\[ \forall a \in X : (a, a) \notin R \]
\[ A = \{1, 2, 3\}, \quad R \subseteq A \times A \]

\[ \{(1, 1), (1, 3), (2, 1), (2, 2), (3, 3)\} \]
\[ A = \{1, 2, 3\}, \quad R \subseteq A \times A \]

\[
\{(1, 1), (1, 3), (2, 1), (2, 2), (3, 3)\}
\]

\[
\{(1, 2), (2, 3), (3, 1)\}
\]
\[ A = \{1, 2, 3\}, R \subseteq A \times A \]

\( \{(1, 1), (1, 3), (2, 1), (2, 2), (3, 3)\} \)

\( \{(1, 2), (2, 3), (3, 1)\} \)

\( \{(1, 2), (2, 2), (2, 3), (3, 1)\} \)
\( R \subseteq X \times X \)

**Definition (Symmetric)**

\[ \forall a, b \in X : aRb \implies bRa \]

\[ a \quad \rightarrow \quad b \]

\[ b \quad \rightarrow \quad a \]
Definition (Symmetric)

\[ \forall a, b \in X : aRb \implies bRa \]

Definition (AntiSymmetric)

\[ \forall a, b \in X : (aRb \land bRa) \implies a = b \]
Definition (Symmetric)

\[ \forall a, b \in X : aRb \implies bRa \]

Definition (AntiSymmetric)

\[ \forall a, b \in X : (aRb \land bRa) \implies a = b \]
$R \subseteq X \times X$

**Definition (Symmetric)**

$\forall a, b \in X : aRb \implies bRa$

![Diagram showing symmetric relation between a and b]

**Definition (AntiSymmetric)**

$\forall a, b \in X : (aRb \land bRa) \implies a = b$

> *is antisymmetric.*
\[ A = \{1, 2, 3\}, \quad R \subseteq A \times A \]

\[ \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1), (3, 3)\} \]
\[ A = \{1, 2, 3\}, \quad R \subseteq A \times A \]

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\[ \{(1, 1), (2, 2), (3, 3)\}\]
$A = \{1, 2, 3\}, \quad R \subseteq A \times A$

$\{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1), (3, 3)\}$

$\{(1, 2), (2, 3), (2, 2), (3, 1)\}$

$\{(1, 1), (2, 2), (3, 3)\}$

$\{(1, 2), (2, 1), (2, 3)\}$
\[ R \subseteq X \times X \]

**Definition (Transitive)**

\[ \forall a, b, c \in X : aRb \land bRc \implies aRc \]
\[ A = \{1, 2, 3\}, R \subseteq A \times A \]

\[ \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\} \]
$A = \{1, 2, 3\}, R \subseteq A \times A$

$\{ (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3) \}$

$\{ (1, 2), (2, 3), (3, 1) \}$
\[ A = \{1, 2, 3\}, \quad R \subseteq A \times A \]

\[
\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}
\]

\[
\{(1, 2), (2, 3), (3, 1)\}
\]

\[
\{(1, 3)\}
\]
$A = \{1, 2, 3\}, R \subseteq A \times A$

$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$

$\{(1, 2), (2, 3), (3, 1)\}$

$\{(1, 3)\}$

$\emptyset$
\[ R \subseteq X \times X \]

**Definition (Connex)**

\[ \forall a, b \in X : aRb \lor bRa \]
$R \subseteq X \times X$

**Definition (Connex)**

$\forall a, b \in X : aRb \lor bRa$

**Definition (Trichotomous)**

$\forall a, b \in X : \text{exactly one of } aRb, bRa, \text{ or } a = b \text{ holds}$
Theorem

\[ R \text{ is reflexive } \iff I \subseteq R \]

\[ I = \{(a, a) \in A \times A \mid a \in A\} \]
Theorem

$R$ is reflexive $\iff I \subseteq R$

$I = \{(a, a) \in A \times A \mid a \in A\}$

Theorem

$R$ is symmetric $\iff R^{-1} = R$
Theorem

\[ R \text{ is reflexive} \iff I \subseteq R \]

\[ I = \{(a, a) \in A \times A \mid a \in A\} \]

Theorem

\[ R \text{ is symmetric} \iff R^{-1} = R \]

Theorem

\[ R \text{ is transitive} \iff R \circ R \subseteq R \]
Theorem

\[ R \text{ is reflexive } \iff I \subseteq R \]

\[ I = \{(a, a) \in A \times A \mid a \in A\} \]

Theorem

\[ R \text{ is symmetric } \iff R^{-1} = R \]

Theorem

\[ R \text{ is transitive } \iff R \circ R \subseteq R \]

\[(1, 2), (2, 3), (1, 3), (4, 4)\]
Equivalence Relations
Definition (Equivalence Relation)

\( R \) is an equivalence relation on \( X \) iff \( R \) is

- reflexive
- symmetric
- transitive
Definition (Equivalence Relation)

$R$ is an *equivalence relation* on $X$ iff $R$ is

- reflexive
- symmetric
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$$a \sim b \iff a \% 12 = b \% 12$$

Why are equivalence relations important?
Definition (Equivalence Relation)

$R$ is an *equivalence relation* on $X$ iff $R$ is

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\[ = \in R \times R \]

\[ \parallel \in L \times L \]

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Definition (Equivalence Relation)

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- symmetric
- transitive

\[
\begin{align*}
&\, = \in R \times R \\
&\, \| \in L \times L \\
\Rightarrow &\, a \sim b \iff a \% 12 = b \% 12
\end{align*}
\]

Why are equivalence relations important?
Equivalence Relations as Abstractions
Equivalence Relations as Abstractions
Equivalence Relations as Abstractions

“全国人民代表大会各省代表团”
Equivalence Relations as Abstractions

Equivalence Relation $\iff$ Partition

“全国人民代表大会各省代表团”
Partition

“不空、不漏、不重”
Definition (Partition)

A family of sets \( \{A_\alpha : \alpha \in I\} \) is a partition of \( X \) if

(i) \[
\forall \alpha \in I : A_\alpha \neq \emptyset
\]

(ii) \[
\bigcup_{\alpha \in I} A_\alpha = X
\]

(iii) \[
\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta
\]
Definition (Partition)

A family of sets \( \{A_\alpha : \alpha \in I\} \) is a partition of \( X \) if

(i)

\[
\forall \alpha \in I : A_\alpha \neq \emptyset
\]

\[
(\forall \alpha \in I \exists x \in X : x \in A_\alpha)
\]

(ii)

\[
\bigcup_{\alpha \in I} A_\alpha = X
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(iii)

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\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta
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\]

\[
(\forall x \in X \exists \alpha \in I : x \in A_\alpha)\]

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\]

\[(\forall x \in X \exists \alpha \in I : x \in A_\alpha)\]

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\[
\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta
\]

\[(\forall \alpha, \beta \in I : A_\alpha \cap A_\beta \neq \emptyset \implies A_\alpha = A_\beta)\]
Equivalence Relation $R \subseteq X \times X \implies$ Partition $\Pi$ of $X$
Equivalence Relation $R \subseteq X \times X$ $\implies$ Partition $\Pi$ of $X$

Definition (Equivalence Class)
The *equivalence class of a modulo* $R$ is a set:

$$[a]_R = \{ b \in X : aRb \}$$
Equivalence Relation $R \subseteq X \times X \implies$ Partition $\Pi$ of $X$

**Definition (Equivalence Class)**

The *equivalence class of a modulo* $R$ is a set:

$$[a]_R = \{ b \in X : aRb \}$$

**Definition (Quotient Set)**

The *quotient set* is a set:

$$X/R = \{ [a]_R \mid a \in X \}$$
Theorem

\[ X/R = \{ [a]_R \mid a \in X \} \text{ is a partition of } X. \]
Theorem

\[ X/R = \{ [a]_R \mid a \in X \} \text{ is a partition of } X. \]

\[ \forall a \in X : [a]_R \neq \emptyset \]
Theorem

\[ X/R = \{ [a]_R \mid a \in X \} \text{ is a partition of } X. \]

\[ \forall a \in X : [a]_R \neq \emptyset \]

\[ \forall a \in X : \exists b \in X : a \in [b]_R \]
Theorem

\[ X/R = \{[a]_R \mid a \in X\} \text{ is a partition of } X. \]

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Theorem

\[ \forall a \in X, b \in X : [a]_R \cap [b]_R = \emptyset \lor [a]_R = [b]_R \]
Theorem

\[ X/R = \{ [a]_R \mid a \in X \} \text{ is a partition of } X. \]

\[ \forall a \in X : [a]_R \neq \emptyset \]

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Theorem

\[ \forall a \in X, b \in X : [a]_R \cap [b]_R = \emptyset \vee [a]_R = [b]_R \]

\[ \forall a \in X, b \in X : [a]_R \cap [b]_R \neq \emptyset \implies [a]_R = [b]_R \]
Partition $\Pi$ of $X$ $\Rightarrow$ Equivalence Relation $R \subseteq X \times X$
Partition $\Pi$ of $X \implies$ Equivalence Relation $R \subseteq X \times X$

**Definition**

$$(a, b) \in R \iff \exists S \in \Pi : a \in S \land b \in S$$

Theorem $R$ is an equivalence relation on $X$.

$\forall x \in X : xRx$

$\forall x, y \in X : xRy \implies yRx$

$\forall x, y, z \in X : xRy \land yRz \implies xRz$
Partition $\Pi$ of $X \implies$ Equivalence Relation $R \subseteq X \times X$

**Definition**

$$(a, b) \in R \iff \exists S \in \Pi : a \in S \land b \in S$$

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Partition $\Pi$ of $X \implies$ Equivalence Relation $R \subseteq X \times X$

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$$(a, b) \in R \iff \exists S \in \Pi : a \in S \land b \in S$$

$$R = \{(a, b) \in X \times X \mid \exists S \in \Pi : a \in S \land b \in S\}$$

**Theorem**

$R$ is an equivalence relation on $X$. 

Partition $\Pi$ of $X \implies$ Equivalence Relation $R \subseteq X \times X$

Definition

$$(a, b) \in R \iff \exists S \in \Pi : a \in S \land b \in S$$

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$R$ is an equivalence relation on $X$.

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Partition $\Pi$ of $X \implies$ Equivalence Relation $R \subseteq X \times X$

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**Theorem**

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$$\forall x \in X : xRx$$

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$$\forall x, y, z \in X : xRy \land yRz \implies xRz$$
Equivalence Relation $\iff$ Partition
Definition

\[ \sim \subseteq \mathbb{N} \times \mathbb{N} \]

\[(a, b) \sim (c, d) \iff a +_\mathbb{N} d = b +_\mathbb{N} c \]
Definition

\[ \sim \subseteq \mathbb{N} \times \mathbb{N} \]

\[(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_{\mathbb{N}} c\]

Theorem

\(\sim\) is an equivalence relation.
Definition

\[ \sim \subseteq \mathbb{N} \times \mathbb{N} \]

\[ (a, b) \sim (c, d) \iff a +_\mathbb{N} d = b +_\mathbb{N} c \]

Theorem

\[ \sim \text{ is an equivalence relation.} \]

\[ Q: \text{What is } \mathbb{N} \times \mathbb{N}/\sim? \]
Definition

\[ \sim \subseteq \mathbb{N} \times \mathbb{N} \]

\[(a, b) \sim (c, d) \iff a +_\mathbb{N} d = b +_\mathbb{N} c\]

Theorem

\[\sim\] is an equivalence relation.

Q: What is \( \mathbb{N} \times \mathbb{N}/\sim \)?

Definition (\(\mathbb{Z}\))

\[\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N}/\sim\]
Definition

\[ \sim \subseteq \mathbb{N} \times \mathbb{N} \]

\[(a, b) \sim (c, d) \iff a +_\mathbb{N} d = b +_\mathbb{N} c \]

Theorem

\(\sim \) is an equivalence relation.

\[ Q : \text{What is } \mathbb{N} \times \mathbb{N}/\sim ? \]

Definition (\(\mathbb{Z}\))

\[ \mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N}/\sim \]

\[ [(1, 3)]_\sim = \{(0, 2), (1, 3), (2, 4), (3, 5), \ldots \} \triangleq -2 \in \mathbb{Z} \]
\[ \mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim \]
Definition ($+\mathbb{Z}$)

\[(m_1, n_1) + \mathbb{Z} (m_2, n_2) = [m_1 + \mathbb{N} m_2, n_1 + \mathbb{N} n_2]\]
Definition ($+\mathbb{Z}$)

\[ [(m_1, n_1)] +_{\mathbb{Z}} [(m_2, n_2)] = [m_1 +_{\mathbb{N}} m_2, n_1 +_{\mathbb{N}} n_2] \]

Definition ($\cdot_{\mathbb{Z}}$)

\[ [(m_1, n_1)] \cdot_{\mathbb{Z}} [(m_2, n_2)] = [m_1 \cdot_{\mathbb{N}} m_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} n_2, m_1 \cdot_{\mathbb{N}} n_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} m_2] \]
Definition

\[ \sim \subseteq \mathbb{Z} \times \mathbb{Z} \setminus \{0\} \]

\[(a, b) \sim (c, d) \iff a \cdot \mathbb{Z} d = b \cdot \mathbb{Z} c\]
Definition

\[ \sim \subseteq \mathbb{Z} \times \mathbb{Z} \setminus \{0\} \]

\[(a, b) \sim (c, d) \iff a \cdot \mathbb{Z} d = b \cdot \mathbb{Z} c\]

Definition \((\mathbb{Q})\)

\[ \mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim \]
\[ \mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim \]
How to define $\mathbb{R}$ as equivalence classes of ordered pairs of $\mathbb{Q}$?
How to define $\mathbb{R}$ as equivalence classes of ordered pairs of $\mathbb{Q}$?
Thank You!