问题与讨论

2014-11-12

- 25.1-4/5/10
- 25.2-4
- 25.3-2/3
- 25-2

25.1-4

Show that matrix multiplication defined by EXTEND-SHORTEST-PATHS is associative.

25.1-5

Show how to express the single-source shortest-paths problem as a product of matrices and a vector. Describe how evaluating this product corresponds to a Bellman-Ford-like algorithm (see Section 24.1).

```
EXTEND-SHORTEST-PATHS(L, W)
n = L.rows
let L' = (l'_ij) be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        l'_ij = ∞
        for k = 1 to n
            l'_ij = min(l'_ij, l_ik + w_kj)
return L'
```

```
SLOW-ALL-PAIRS-SHORTEST-PATHS(W)
n = W.rows
L^(1) = W
for m = 2 to n - 1
    let L^(m) be a new n × n matrix
    L^(m) = EXTEND-SHORTEST-PATHS(L^(m-1), W)
return L^(n-1)
```

25.1-10

Give an efficient algorithm to find the length (number of edges) of a minimum-length negative-weight cycle in a graph.

25.2-4

As it appears above, the Floyd-Warshall algorithm requires $\Theta(n^3)$ space, since we compute $d_{ij}^{(k)}$ for $i, j, k = 1, 2, \ldots, n$. Show that the following procedure, which simply drops all the superscripts, is correct, and thus only $\Theta(n^2)$ space is required.

```
FLOYD-WARSHALL'(W)
n = W.rows
D = W
for k = 1 to n
    for i = 1 to n
        for j = 1 to n
            d_ij = min(d_ij, d_ik + d_kj)
return D
```

$$
\begin{aligned}
d_{ij}^{(k)} &\leftarrow \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k)} + d_{kj}^{(k-1)}\right) \\
d_{ij}^{(k)} &\leftarrow \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k)}\right) \\
d_{ij}^{(k)} &\leftarrow \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k)} + d_{kj}^{(k)}\right)
\end{aligned}
$$

25.3-2

What is the purpose of adding the new vertex s to V, yielding V'?

25.3-3

Suppose that $w(u, v) \geq 0$ for all edges $(u, v) \in E$. What is the relationship between the weight functions w and $\hat{w}$?

25-2 *Shortest paths in ϵ-dense graphs*

A graph $G = (V, E)$ is ***ϵ-dense*** if $|E| = \Theta(V^{1+\epsilon})$ for some constant ϵ in the range $0 < \epsilon \leq 1$. By using d-ary min-heaps (see Problem 6-2) in shortest-paths algorithms on ϵ-dense graphs, we can match the running times of Fibonacci-heap-based algorithms without using as complicated a data structure.

a. What are the asymptotic running times for INSERT, EXTRACT-MIN, and DECREASE-KEY, as a function of d and the number n of elements in a d-ary min-heap? What are these running times if we choose $d = \Theta(n^{\alpha})$ for some constant $0 < \alpha \leq 1$? Compare these running times to the amortized costs of these operations for a Fibonacci heap.

6-2 *Analysis of d-ary heaps*

A ***d-ary heap*** is like a binary heap, but (with one possible exception) non-leaf nodes have d children instead of 2 children.

a. How would you represent a d-ary heap in an array?

b. What is the height of a d-ary heap of n elements in terms of n and d?

c. Give an efficient implementation of EXTRACT-MAX in a d-ary max-heap. Analyze its running time in terms of d and n.

d. Give an efficient implementation of INSERT in a d-ary max-heap. Analyze its running time in terms of d and n.

e. Give an efficient implementation of INCREASE-KEY(A, i, k), which flags an error if $k < A[i]$, but otherwise sets $A[i] = k$ and then updates the d-ary max-heap structure appropriately. Analyze its running time in terms of d and n.

Procedure	Binary heap (worst-case)	Fibonacci heap (amortized)
MAKE-HEAP	$\Theta(1)$	$\Theta(1)$
INSERT	$\Theta(\lg n)$	$\Theta(1)$
MINIMUM	$\Theta(1)$	$\Theta(1)$
EXTRACT-MIN	$\Theta(\lg n)$	$O(\lg n)$
UNION	$\Theta(n)$	$\Theta(1)$
DECREASE-KEY	$\Theta(\lg n)$	$\Theta(1)$
DELETE	$\Theta(\lg n)$	$O(\lg n)$

b. Show how to compute shortest paths from a single source on an ϵ-dense directed graph $G = (V, E)$ with no negative-weight edges in $O(E)$ time. (*Hint:* Pick d as a function of ϵ.)

c. Show how to solve the all-pairs shortest-paths problem on an ϵ-dense directed graph $G = (V, E)$ with no negative-weight edges in $O(VE)$ time.

d. Show how to solve the all-pairs shortest-paths problem in $O(VE)$ time on an ϵ-dense directed graph $G = (V, E)$ that may have negative-weight edges but has no negative-weight cycles.