Problem Solving

2-3 Counting

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Institute of Computer Software

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CS1.2-1

In how many ways can we pass out k distinct pieces of fruit to n children (with no restriction on how many pieces of fruit a child may get)?

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THE TWELVEFOLD WAY

balls per urn	unrestricted	≤ 1	≥ 1
n labeled balls, m labeled urns	n-tuples of m things	n-permutations of m things	partitions of $\{1, \ldots, n\}$ into m ordered parts
n unlabeled balls, m labeled urns	n-multicombinations of m things	n-combinations of m things	compositions of n into m parts
n labeled balls, m unlabeled urns	partitions of $\{1, \ldots, n\}$ into $\leq m$ parts	n pigeons into m holes	partitions of $\{1, \ldots, n\}$ into m parts
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$k \leq n$: $n^{\underline{k}}$

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$k > n$: Zero!

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CS1.2-15

A tennis club has $2n$ members. We want to pair up the members by twos for singles matches.

(1) In how many ways can we pair up all the members of the club?

(2) Suppose that in addition to specifying who plays whom, we also determine who serves first for each pairing. Now in how many ways can we specify our pairs?

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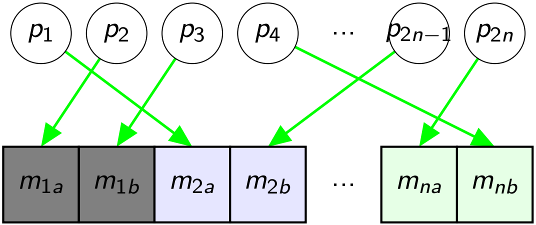

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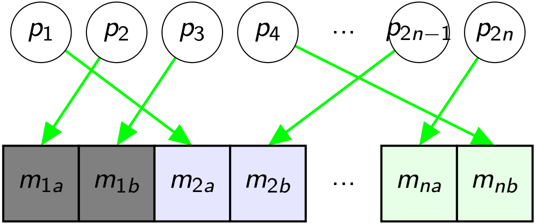

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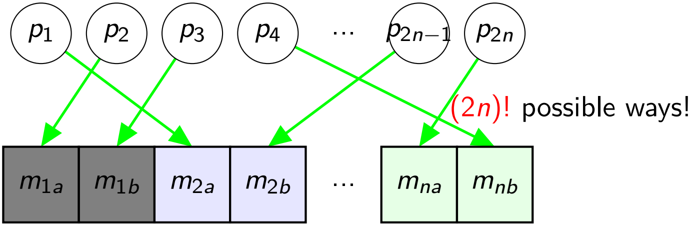

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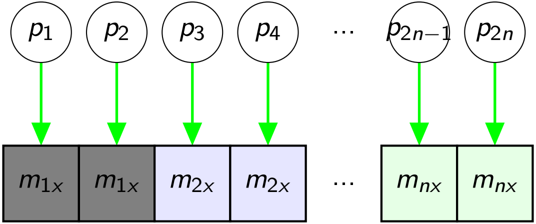

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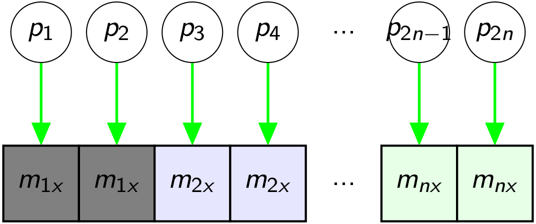

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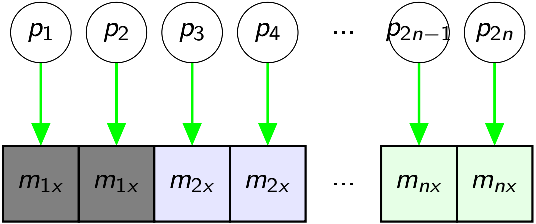

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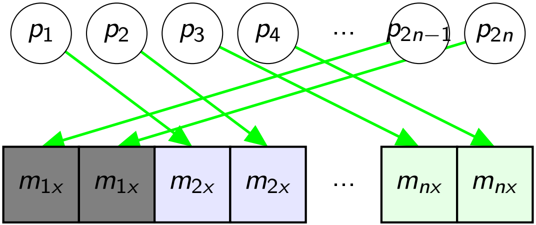

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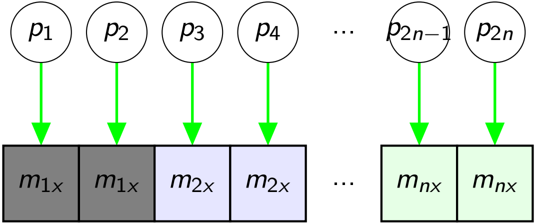

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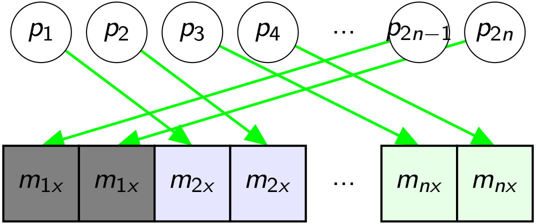

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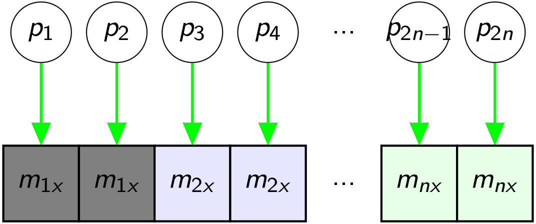

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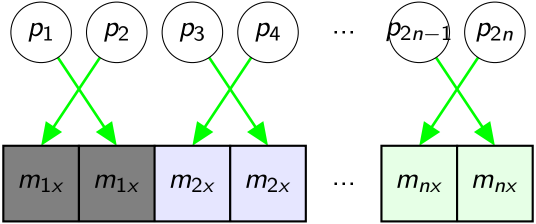

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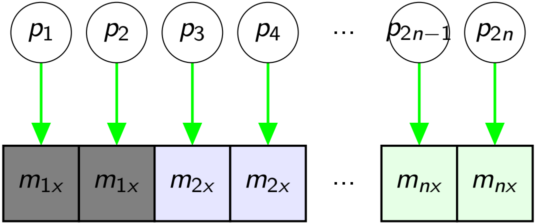

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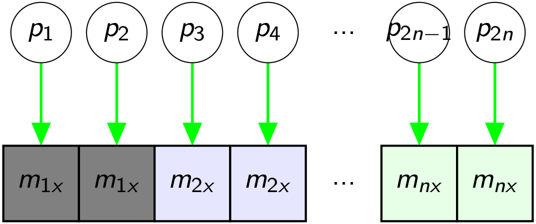
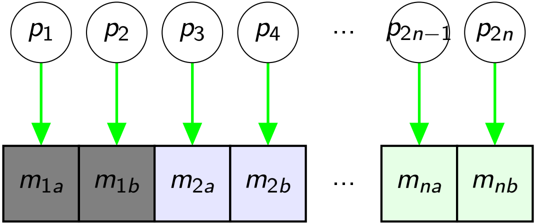


How many ways are equivalent to this? $n! \times 2^n$

So, the number of different ways is $\frac{(2n)!}{n! \times 2^n} = \frac{(2n)^{\underline{n}}}{2^n}$!

Question-2

Suppose that in addition to specifying who plays whom, we also determine who serves first for each pairing. Now in how many ways can we specify our pairs?



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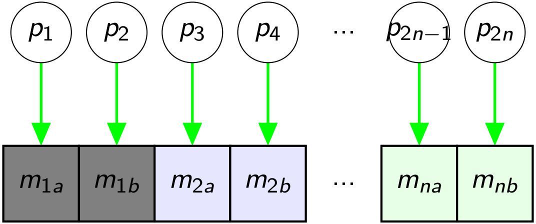

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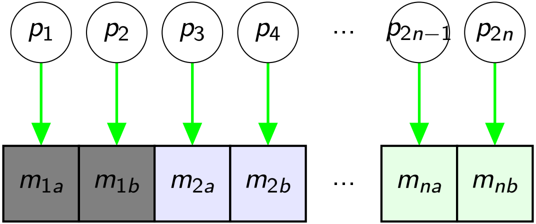

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So, the number of different ways is $\frac{(2n)!}{n!} = (2n)^{\underline{n}}!$

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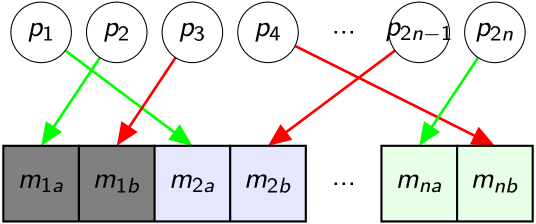

Step1: select n players from the $2n$ players to play the n as for all matches, totally $\binom{2n}{n}$.

Step2: assign the remaining n players to the n bs.
Same as putting n different balls into n different bins, $n^{\underline{n}}$

So, there are $\binom{2n}{n} \times n^{\underline{n}}$ different ways in total.

CS 1.5-4

Use multisets to determine the number of ways to pass out n identical apples to m children. Assume that a child may get more than one apple.

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Theorem

The # of weak composition of n with m terms is

$$\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$$

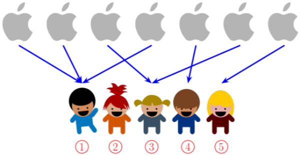
Theorem (CS Theorem 1.8)

The # of n-element multisets chosen from an m-element set is

$$\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$$

$n = 7 \quad m = 5$ (Apples and Children)

$\{1, \; 1, \; 3, \; 1, \; 4, \; 3, \; 5\}$



$7 = 3 + 0 + 2 + 1 + 1$

CS 1.5-12

A standard notation for the number of partitions of an n-element set into k classes is $S(n, k)$. Because the empty family of subsets of the empty set is a partition of the empty set, $S(0, 0)$ is 1. In addition, $S(n, 0)$ is 0 for $n > 0$, because there are no partitions of a nonempty set into no parts. $S(1, 1)$ is 1.

1. Explain why $S(n, n)$ is 1 for all $n > 0$. Explain why $S(n, 1)$ is 1 for all $n > 0$.
2. Explain why $S(n, k) = S(n-1, k-1) + kS(n-1, k)$ for $1 < k < n$.
3. Make a table like Table 1.1 that shows the values of $S(n, k)$ for values of n and k ranging from 1 to 6.

Set Partition (CS : 1.5 – 12)

$$\left\{ {n \atop k} \right\} : \ \# \text{ of partitions of set } N \text{ into } k \text{ parts}$$

$$\left\{ {n \atop k} \right\} = \underbrace{\left\{ {n-1 \atop k-1} \right\}}_{n \text{ is alone}} + \underbrace{k \left\{ {n-1 \atop k} \right\}}_{n \text{ is not alone}} \qquad (n > 0, k > 0)$$

选做题 (Summation)

请计算如下代码段的返回值 r。

```
procedure CONUNDRUM(n)
    r ← 0

    for i ← 1 to n do
        for j ← i + 1 to n do
            for k ← i + j − 1 to n do
                r ← r + 1
            end for
        end for
    end for

    return r

end procedure
```

$$\# = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1$$

选做题 (Summation)

$\# = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1$

$$
\begin{aligned}
\# &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 \\
&= \sum_{i=1}^{n} \sum_{j=i+1}^{n-i+1} (n-i+2-j) \\
&= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left((n-i+2)(n-i+1-(i+1)+1) - \frac{(i+1+n-i+1)(n-i+1-(i+1)+1)}{2} \right) \\
&= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left((n-i+2)(n-2i+1) - \frac{(n+2)(n-2i+1)}{2} \right) \\
&= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{2(n-i+2)-(n+2)}{2} (n-2i+1) \right) \\
&= \frac{\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n-2i+2)(n-2i+1)}{2} \\
&= \frac{\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n+1)(n+2) + 4 \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} i^2 - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (4n+6)i}{2} \\
&= \frac{\lfloor \frac{n}{2} \rfloor (n+1)(n+2) + 4 \cdot \frac{\lfloor \frac{n}{2} \rfloor \left(\lfloor \frac{n}{2} \rfloor + 1 \right) \left(2 \lfloor \frac{n}{2} \rfloor + 1 \right)}{6} - \frac{\lfloor \frac{n}{2} \rfloor \left(\lfloor \frac{n}{2} \rfloor + 1 \right) (4n+6)}{2}}{2}
\end{aligned}
$$

选做题 (Summation)

$\# = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1$

if n is even if n is even

$$\begin{aligned}
\# &= \frac{\lfloor \frac{n}{2} \rfloor (n+1)(n+2) + 4 \cdot \frac{\lfloor \frac{n}{2} \rfloor \left(\lfloor \frac{n}{2} \rfloor + 1 \right) \left(2 \lfloor \frac{n}{2} \rfloor + 1 \right)}{6} - \frac{\lfloor \frac{n}{2} \rfloor \left(\lfloor \frac{n}{2} \rfloor + 1 \right)(4n+6)}{2}}{2} \\
&= \frac{\frac{n}{2}(n+1)(n+2) + 4 \cdot \frac{\frac{n}{2}\left(\frac{n}{2}+1\right)\left(2\frac{n}{2}+1\right)}{6} - \frac{\frac{n}{2}\left(\frac{n}{2}+1\right)(4n+6)}{2}}{2} \\
&= \frac{n(n+2)(2n-1)}{24}
\end{aligned}$$

选做题 (Summation)

$\# = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1$

if n is even if n is odd

$$\# = \frac{\lfloor \frac{n}{2} \rfloor (n+1)(n+2) + 4 \cdot \frac{\lfloor \frac{n}{2} \rfloor \left(\lfloor \frac{n}{2} \rfloor + 1 \right) \left(2 \lfloor \frac{n}{2} \rfloor + 1 \right)}{6} - \frac{\lfloor \frac{n}{2} \rfloor \left(\lfloor \frac{n}{2} \rfloor + 1 \right) (4n+6)}{2}}{2}$$

$$= \frac{\frac{n-1}{2} (n+1)(n+2) + 4 \cdot \frac{\frac{n-1}{2} \left(\frac{n-1}{2} + 1 \right) \left(2 \frac{n-1}{2} + 1 \right)}{6} - \frac{\frac{n-1}{2} \left(\frac{n-1}{2} + 1 \right) (4n+6)}{2}}{2}$$

$$= \frac{(n-1)(n+1)(2n+3)}{24}$$

选做题 (Summation)

```
n=1, r=0, a_1-a_0=0
n=2, r=1, a_2-a_1=1
n=3, r=3, a_3-a_2=2
n=4, r=7, a_4-a_3=4
n=5, r=13, a_5-a_4=6
n=6, r=22, a_6-a_5=9
n=7, r=34, a_7-a_6=12
n=8, r=50, a_8-a_7=16
n=9, r=70, a_9-a_8=20
n=10, r=95, a_10-a_9=25
n=11, r=125, a_11-a_10=30
n=12, r=161, a_12-a_11=36
n=13, r=203, a_13-a_12=42
n=14, r=252, a_14-a_13=49
n=15, r=308, a_15-a_14=56
n=16, r=372, a_16-a_15=64
n=17, r=444, a_17-a_16=72
n=18, r=525, a_18-a_17=81
n=19, r=615, a_19-a_18=90
n=20, r=715, a_20-a_19=100
n=21, r=825, a_21-a_20=110
n=22, r=946, a_22-a_21=121
n=23, r=1078, a_23-a_22=132
n=24, r=1222, a_24-a_23=144
n=25, r=1378, a_25-a_24=156
n=26, r=1547, a_26-a_25=169
n=27, r=1729, a_27-a_26=182
n=28, r=1925, a_28-a_27=196
n=29, r=2135, a_29-a_28=210
n=30, r=2360, a_30-a_29=225
n=31, r=2600, a_31-a_30=240
```

因为三重循环，所以 r 为关于 n 的三次式。
打表代入数据算出系数
观察到奇偶有差别，分奇偶讨论：

$$r_n = \begin{cases} \frac{1}{12}n^3 + \frac{1}{8}n^2 - \frac{1}{12}n & , n \text{ is even} \\ \frac{1}{12}n^3 + \frac{1}{8}n^2 - \frac{1}{12}n - \frac{1}{8} & , n \text{ is odd} \end{cases}$$

选做题 (Summation)

```
n=1, r=0, a_1-a_0=0
n=2, r=1, a_2-a_1=1
n=3, r=3, a_3-a_2=2
n=4, r=7, a_4-a_3=4
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n=7, r=34, a_7-a_6=12
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$$r_n = \frac{1}{12}n^3 + \frac{1}{8}n^2 - \frac{1}{12}n - \frac{1}{16}\left(1 - (-1)^n\right)$$

Thank You!

Questions?

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