3-12 Matching & Covers

Jun Ma

majun@nju.edu.cn

December 10, 2020

CZ 8.3

CZ 8.3

Figure 8.5 shows two bipartite graphs G_1 and G_2, each with partite sets $U = \{v, w, x, y, z\}$ and $W = \{a, b, c, d, e\}$. In each case, can U be matched to W?

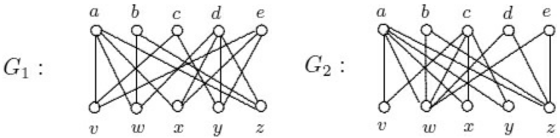


Figure 8.5: The graphs G_1 and G_2 in Exercise 8.3

CZ 8.3

Figure 8.5 shows two bipartite graphs G_1 and G_2, each with partite sets $U = \{v, w, x, y, z\}$ and $W = \{a, b, c, d, e\}$. In each case, can U be matched to W?

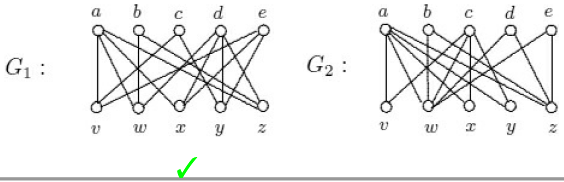


Figure 8.5: The graphs G_1 and G_2 in Exercise 8.3

CZ 8.3

Figure 8.5 shows two bipartite graphs G_1 and G_2, each with partite sets $U = \{v, w, x, y, z\}$ and $W = \{a, b, c, d, e\}$. In each case, can U be matched to W?

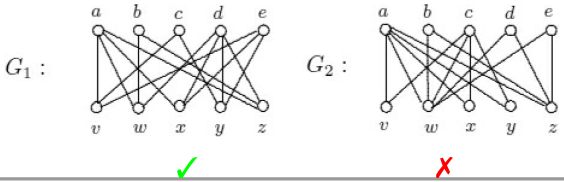


Figure 8.5: The graphs G_1 and G_2 in Exercise 8.3

CZ 8.3

Figure 8.5 shows two bipartite graphs G_1 and G_2, each with partite sets $U = \{v, w, x, y, z\}$ and $W = \{a, b, c, d, e\}$. In each case, can U be matched to W?

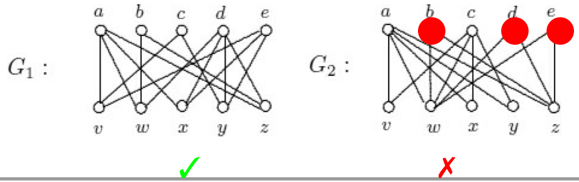


Figure 8.5: The graphs G_1 and G_2 in Exercise 8.3

CZ 8.5

Prove that every tree has **at most one** perfect matching.

CZ 8.5

Prove that every tree has **at most one** perfect matching.

Proof.

- If $|T| = 2k + 1$, there is no perfect matching.

CZ 8.5

Prove that every tree has **at most one** perfect matching.

Proof.

- If $|T| = 2k + 1$, there is no perfect matching. ✓

CZ 8.5

Prove that every tree has **at most one** perfect matching.

Proof.

- If $|T| = 2k + 1$, there is no perfect matching. ✓
- If $|T| = 2k$, prove by introduction on k.
  - **B:** $k = 1$, obviously holds
  - **H:** assume the property hold for all $k < m$
  - **I:** $k = m$

CZ 8.5

Prove that every tree has **at most one** perfect matching.

Proof.

- If $|T| = 2k + 1$, there is no perfect matching. ✓
- If $|T| = 2k$, prove by introduction on k.
  - **B:** $k = 1$, obviously holds
  - **H:** assume the property hold for all $k < m$
  - **I:** $k = m$

CZ 8.5

Prove that every tree has **at most one** perfect matching.

Proof.

- If $|T| = 2k + 1$, there is no perfect matching. ✓
- If $|T| = 2k$, prove by introduction on k.
  - **B:** $k = 1$, obviously holds
  - **H:** assume the property hold for all $k < m$
  - **I:** $k = m$
    - T is a tree $\Rightarrow$ there must be at least one vertex $v \in T$ s.t. $\deg v = 1$. Assume $(u, v) \in T.E$. If T has a perfect matching M, $(u, v) \in M$

CZ 8.5

Prove that every tree has **at most one** perfect matching.

Proof.

- If $|T| = 2k+1$, there is no perfect matching. ✓
- If $|T| = 2k$, prove by introduction on k.
  - **B:** $k = 1$, obviously holds
  - **H:** assume the property hold for all $k < m$
  - **I:** $k = m$
    - T is a tree $\Rightarrow$ there must be at least one vertex $v \in T$ s.t. $\deg v = 1$. Assume $(u, v) \in T.E$. If T has a perfect matching M, $(u, v) \in M$
    - $T - \{u, v\}$ would generate a set of component $\{T_1, T_2, ..., T_x\}$ $(x \geq 1)$

CZ 8.5

Prove that every tree has **at most one** perfect matching.

Proof.

- If $|T| = 2k + 1$, there is no perfect matching. ✓
- If $|T| = 2k$, prove by introduction on k.
  - **B:** $k = 1$, obviously holds
  - **H:** assume the property hold for all $k < m$
  - **I:** $k = m$
    - T is a tree $\Rightarrow$ there must be at least one vertex $v \in T$ s.t. $\deg v = 1$. Assume $(u, v) \in T.E$. If T has a perfect matching M, $(u, v) \in M$
    - $T - \{u, v\}$ would generate a set of component $\{T_1, T_2, ..., T_x\}$ $(x \geq 1)$
    - If the order of any T_i is odd, then T has no perfect matching.

CZ 8.5

Prove that every tree has **at most one** perfect matching.

Proof.

- If $|T| = 2k + 1$, there is no perfect matching. ✓
- If $|T| = 2k$, prove by introduction on k.
  - **B:** $k = 1$, obviously holds
  - **H:** assume the property hold for all $k < m$
  - **I:** $k = m$
    - T is a tree $\Rightarrow$ there must be at least one vertex $v \in T$ s.t. $\deg v = 1$. Assume $(u, v) \in T.E$. If T has a perfect matching M, $(u, v) \in M$
    - $T - \{u, v\}$ would generate a set of component $\{T_1, T_2, ..., T_x\}$ $(x \geq 1)$
    - If the order of any T_i is odd, then T has no perfect matching. ✓

CZ 8.5

Prove that every tree has **at most one** perfect matching.

Proof.

- If $|T| = 2k + 1$, there is no perfect matching. ✓
- If $|T| = 2k$, prove by introduction on k.
  - **B:** $k = 1$, obviously holds
  - **H:** assume the property hold for all $k < m$
  - **I:** $k = m$
    - T is a tree $\Rightarrow$ there must be at least one vertex $v \in T$ s.t. $\deg v = 1$. Assume $(u, v) \in T.E$. If T has a perfect matching M, $(u, v) \in M$
    - $T - \{u, v\}$ would generate a set of component $\{T_1, T_2, ..., T_x\}$ $(x \geq 1)$
    - If the order of any T_i is odd, then T has no perfect matching. ✓
    - Otherwise, each T_i has even order. And by **H**, we could find **exactly one** perfect matching M_i for each T_i.

CZ 8.5

Prove that every tree has **at most one** perfect matching.

Proof.

- If $|T| = 2k + 1$, there is no perfect matching. ✓
- If $|T| = 2k$, prove by introduction on k.
  - **B:** $k = 1$, obviously holds
  - **H:** assume the property hold for all $k < m$
  - **I:** $k = m$
    - T is a tree $\Rightarrow$ there must be at least one vertex $v \in T$ s.t. $\deg v = 1$. Assume $(u, v) \in T.E$. If T has a perfect matching M, $(u, v) \in M$
    - $T - \{u, v\}$ would generate a set of component $\{T_1, T_2, ..., T_x\}$ $(x \geq 1)$
    - If the order of any T_i is odd, then T has no perfect matching. ✓
    - Otherwise, each T_i has even order. And by **H**, we could find **exactly one** perfect matching M_i for each T_i.
    - Then $\{(u, v)\} + \bigcup_{i=1}^{x} M_i$ is one and the only one perfect matching of T.

CZ 8.5

Prove that every tree has **at most one** perfect matching.

Proof.

- If $|T| = 2k + 1$, there is no perfect matching. ✓
- If $|T| = 2k$, prove by introduction on k.
  - **B:** $k = 1$, obviously holds
  - **H:** assume the property hold for all $k < m$
  - **I:** $k = m$
    - T is a tree $\Rightarrow$ there must be at least one vertex $v \in T$ s.t. $\deg v = 1$. Assume $(u, v) \in T.E$. If T has a perfect matching M, $(u, v) \in M$
    - $T - \{u, v\}$ would generate a set of component $\{T_1, T_2, ..., T_x\}$ $(x \geq 1)$
    - If the order of any T_i is odd, then T has no perfect matching. ✓
    - Otherwise, each T_i has even order. And by **H**, we could find **exactly one** perfect matching M_i for each T_i.
    - Then $\{(u, v)\} + \bigcup_{i=1}^{x} M_i$ is one and the only one perfect matching of T. ✓

CZ 8.14

Prove that a graph G without isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.

CZ 8.14

Prove that a graph G without isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.

CZ 8.14

Prove that a graph G without isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.

$\Rightarrow$.

- Let M be a perfect matching of G, then n is even and $\alpha'(G) = |M| = n/2$
- So $\beta'(G) = n - \alpha'(G) = n/2 = |M|$

$\square$

CZ 8.14

Prove that a graph G without isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.

CZ 8.14

Prove that a graph G without isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.

$\Leftarrow$.

- As, $\alpha'(G) + \beta'(G) = n$, and $\alpha'(G) = \beta'(G)$
- n is even and $\alpha'(G) = \beta'(G) = n/2$
- There is an independent edge set (Matching) M consisting of $n/2$ edges, which must be a perfect matching.

□

CZ 8.16

Prove that if G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}$$

CZ 8.16

Prove that if G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}$$

Proof.

CZ 8.16

Prove that if G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}$$

Proof.

- Assume $\beta(G) < \frac{n}{\Delta+1}$, and X be one minimum cover.

CZ 8.16

Prove that if G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}$$

Proof.

- Assume $\beta(G) < \frac{n}{\Delta+1}$, and X be one minimum cover.
- As one vertex $v \in G.V$ could cover at most $\Delta + 1$ vertices (including itself), X could cover at most $|X| \cdot (\Delta + 1)$ vertices, where

$$|X| \cdot (\Delta + 1) = \beta(G) \cdot (\Delta + 1) < n.$$

CZ 8.16

Prove that if G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}$$

Proof.

- Assume $\beta(G) < \frac{n}{\Delta+1}$, and X be one minimum cover.
- As one vertex $v \in G.V$ could cover at most $\Delta + 1$ vertices (including itself), X could cover at most $|X| \cdot (\Delta + 1)$ vertices, where

$$|X| \cdot (\Delta + 1) = \beta(G) \cdot (\Delta + 1) < n. \text{ \textbf{Conflicting!}}$$

□

CZ 8.16

Prove that if G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}$$

CZ 8.16

Prove that if G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}$$

Proof.

- Let C be a vertex cover of G
- $|N(C)| \leq |C| \cdot \Delta$
- $|N(C)| = n - |C|$
- $n - |C| \leq |C| \cdot \Delta$
- So, $|C| \geq \frac{n}{\Delta+1}$
- Finally, $\beta(G) \geq \frac{n}{\Delta+1}$

□

How about $\alpha(G)$

Prove that if G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\alpha(G) \geq \frac{n}{\Delta + 1}$$

How about $\alpha(G)$

Prove that if G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\alpha(G) \geq \frac{n}{\Delta + 1}$$

Proof.

By Construction

How about $\alpha(G)$

Prove that if G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\alpha(G) \geq \frac{n}{\Delta + 1}$$

Proof.

By Construction

To construct an independent set S with $|S| \geq \frac{n}{\Delta+1}$

How about $\alpha(G)$

Prove that if G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\alpha(G) \geq \frac{n}{\Delta + 1}$$

Proof.

By Construction

To construct an independent set S with $|S| \geq \frac{n}{\Delta+1}$

1: **while** $|V(G) > 0|$ **do**
2: Choose $v \in V(G)$
3: $S \leftarrow S \cup \{v\}$
4: $G \leftarrow G - \{v\} - N(v)$

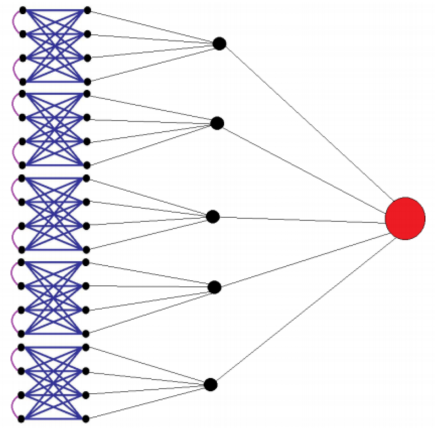
□

CZ 8.18

Give an example of a 5-regular graph that contains no 1-factor.

CZ 8.18

Give an example of a 5-regular graph that contains no 1-factor.



https://math.stackexchange.com/questions/520203/k-regular-simple-graph-without-1-factor

CZ 8.21

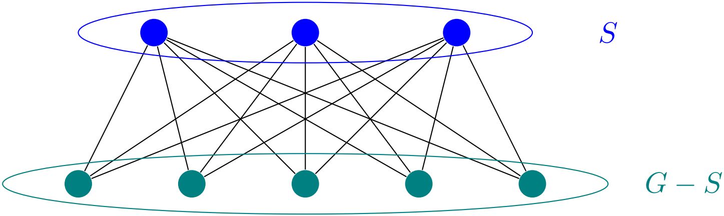
Use Tutte's characterization of graphs with 1-factors (Theorem 8.10) to show that $K_{3,5}$ does not have a 1-factor.

Theorem 8.10 *A graph G contains a 1-factor if and only if* $k_o(G - S) \leq |S|$ *for every proper subset S of* $V(G)$.

CZ 8.21

Use Tutte's characterization of graphs with 1-factors (Theorem 8.10) to show that $K_{3,5}$ does not have a 1-factor.

Theorem 8.10 *A graph G contains a 1-factor if and only if* $k_o(G - S) \leq |S|$ *for every proper subset S of V(G).*



$$k_o(G - S) = 5 > 3 = |S|$$

CZ 8.24

Show that every 3-regular bridgeless graph contains a 2-factor.

CZ 8.24

Show that every 3-regular bridgeless graph contains a 2-factor.

Step-1: Show that every 3-regular bridgeless graph contains a **1-factor**, F.

$\Downarrow$

Step-2: Show that $G - F$ is a 2-factor.

CZ 8.24

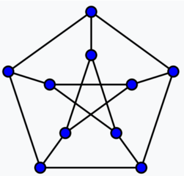
Step-1

Show that every 3-regular bridgeless graph contains a 1-factor.

Theorem (Petersen's theorem)

Every cubic, bridgeless graph contains a perfect matching.

Julius Petersen
(1839 - 1910)



Petersen
Graph

Proof of Petersen's Theorem

Basic Idea

- For every cubic, bridgeless graph $G = (V, E)$ we have that for every set $U \subset V$, $k_o(G - U) \leq |U|$.
- Then by Tutte's theorem, G contains a perfect matching.

Proof of Petersen's Theorem

Proof

- G_i,V_i, m_i
  - G_i: a component with an **odd** number of vertices in the graph induced by the vertex set $V - U$.
  - V_i: the vertices of G_i
  - m_i: the number of edges with one vertex in V_i and one vertex in U.

Proof of Petersen's Theorem

Proof

- G_i,V_i, m_i
  - G_i: a component with an **odd** number of vertices in the graph induced by the vertex set $V - U$.
  - V_i: the vertices of G_i
  - m_i: the number of edges with one vertex in V_i and one vertex in U.
- Then we have

$$\sum_{v \in V_i} \deg_G v = 2|E_i| + m_i$$

  - E_i : the set of edges of G_i with both vertices in V_i

Proof of Petersen's Theorem

Proof

- G_i,V_i, m_i
  - G_i: a component with an **odd** number of vertices in the graph induced by the vertex set $V - U$.
  - V_i: the vertices of G_i
  - m_i: the number of edges with one vertex in V_i and one vertex in U.
- Then we have

$$\sum_{v \in V_i} \deg_G v = 2|E_i| + m_i$$

  - E_i : the set of edges of G_i with both vertices in V_i
-

$$\sum_{v \in V_i} \deg_G v = 3|V_i| \text{ which is odd}$$

Proof of Petersen's Theorem

Proof

- G_i,V_i, m_i
  - G_i: a component with an **odd** number of vertices in the graph induced by the vertex set $V - U$.
  - V_i: the vertices of G_i
  - m_i: the number of edges with one vertex in V_i and one vertex in U.
- Then we have

$$\sum_{v \in V_i} \deg_G v = 2|E_i| + m_i$$

  - E_i : the set of edges of G_i with both vertices in V_i
-

$$\sum_{v \in V_i} \deg_G v = 3|V_i| \text{ which is odd}$$

- m_i must be **odd**, and $m_i \geq 3$ (as G is bridgeless)

Proof of Petersen's Theorem (cont'd)

- **m**: the number of edges in G with one vertex in U and one vertex in the graph induced by $V - U$.

Proof of Petersen's Theorem (cont'd)

- **m**: the number of edges in G with one vertex in U and one vertex in the graph induced by $V - U$.
- Every component with an odd number of vertices contributes **at least** 3 edges to m

Proof of Petersen's Theorem (cont'd)

- **m**: the number of edges in G with one vertex in U and one vertex in the graph induced by $V - U$.
- Every component with an odd number of vertices contributes **at least** 3 edges to m
- So, $k_o(G - V) \leq m/3$.

Proof of Petersen's Theorem (cont'd)

- **m**: the number of edges in G with one vertex in U and one vertex in the graph induced by $V - U$.
- Every component with an odd number of vertices contributes **at least** 3 edges to m
- So, $k_o(G - V) \leq m/3$.
- In the worst case, every edge with one vertex in U contributes to m, and therefore $m \leq 3|U|$

Proof of Petersen's Theorem (cont'd)

- **m**: the number of edges in G with one vertex in U and one vertex in the graph induced by $V - U$.
- Every component with an odd number of vertices contributes **at least** 3 edges to m
- So, $k_o(G - V) \leq m/3$.
- In the worst case, every edge with one vertex in U contributes to m, and therefore $m \leq 3|U|$
- So, $|U| \geq m/3 \geq k_o(G - V)$

Proof of Petersen's Theorem (cont'd)

- **m**: the number of edges in G with one vertex in U and one vertex in the graph induced by $V - U$.
- Every component with an odd number of vertices contributes **at least** 3 edges to m
- So, $k_o(G - V) \leq m/3$.
- In the worst case, every edge with one vertex in U contributes to m, and therefore $m \leq 3|U|$
- So, $|U| \geq m/3 \geq k_o(G - V)$
- By Tutte theorem, G has a 1-factor.

□


Thank
You!