2-4 Recurrences

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Maximal Sum Subarray (Problem $4.1-5$)

- Array $A[1 \cdots n], a_i >=< 0$
- To find (the sum of) an MS in A

$$A[-2, 1, -3, \boxed{4, -1, 2, 1}, -5, 4]$$

MSS$[i]$: the sum of the MS (MS$[i]$) in $A[1 \cdots i]$

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$$\mathsf{mss} = \mathsf{MSS}[n]$$

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mss = MSS$[n]$

Q : Is $a_i \in$ MS$[i]$?

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MSS$[i]$: the sum of the MS *ending with* a_i or 0

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$$\text{mss} = \max_{1 \le i \le n} \text{MSS}[i]$$

Q : where does the MS$[i]$ start?

$\text{MSS}[i]$: the sum of the MS *ending with* a_i or 0

$$\boxed{\text{mss} = \max_{1 \le i \le n} \text{MSS}[i]}$$

Q : where does the $\text{MS}[i]$ start?

$$\boxed{\text{MSS}[i] = \max\{\text{MSS}[i-1] + a_i, 0\}}$$

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$$\boxed{\mathsf{mss} = \max_{1 \le i \le n} \mathsf{MSS}[i]}$$

Q : where does the $\mathsf{MS}[i]$ start?

$$\boxed{\mathsf{MSS}[i] = \max \left\{ \mathsf{MSS}[i-1] + a_i, 0 \right\}}$$

$$\mathsf{MSS}[0] = 0$$

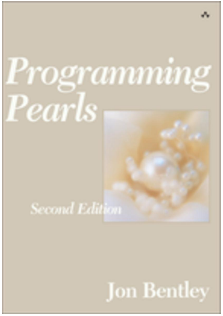
procedure $\text{MSS}(A[1 \cdots n])$

  $\text{MSS}[0] \leftarrow 0$

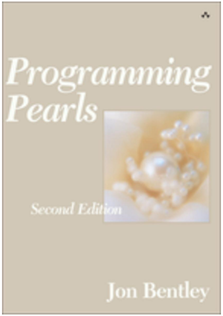
  for $i \leftarrow 1$ **to** n **do**

    $\text{MSS}[i] \leftarrow \max\{\text{MSS}[i-1] + A[i], 0\}$

  return $\max_{1 \le i \le n} \text{MSS}[i]$



Ulf Grenander $O(n^3) \implies O(n^2)$



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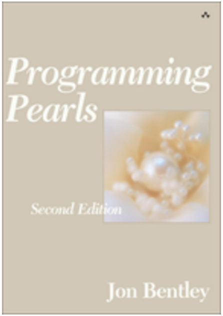

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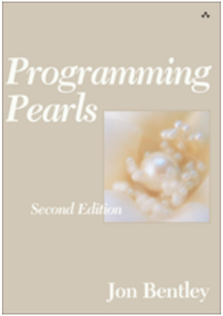

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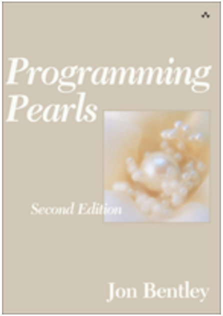

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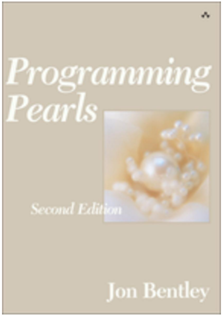

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Michael Shamos Carnegie Mellon seminar

Jay Kadane $O(n)$, ≤ 1 minute

Maximum-product subarray

Maximum-product subarray (Problem 7.4)

- Array $A[1 \ldots n]$
- Find maximum-product subarray of A

Ending with i

		$\frac{1}{2}$	4	-2	5	$-\frac{1}{5}$	8

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MinP$[i]$	1	$\frac{1}{2}$	2	-8	-40	-1	-8

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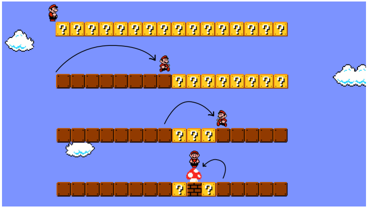
$$\text{MaxP}[i] = \max\{\text{MaxP}[i-1] \cdot a_i, \text{MinP}[i-1] \cdot a_i, a_i\}$$
$$\text{MinP}[i] = \min\{\text{MaxP}[i-1] \cdot a_i, \text{MinP}[i-1] \cdot a_i, a_i\}$$

Binary Search (CLRS $4.5-3$)

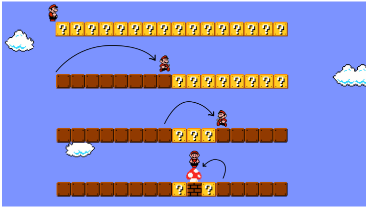
$$T(n) = T(n/2) + \Theta(1)$$

```
procedure BinarySearch(A, L, R, x)
    if R < L then
        return −1

    m ← L + (R − L)/2

    if A[m] = x then
        return m
    else if A[m] > x then
        return BinarySearch(A, L, m − 1, x)
    else
        return BinarySearch(A, m + 1, R, x)
```

$T(n) = \Theta(\log n)$



$$T(n) = \begin{cases} \max\left\{T(\lfloor \frac{n-1}{2} \rfloor), T(\lceil \frac{n-1}{2} \rceil)\right\} + 1, & n > 2 \\ 1, & n = 1 \end{cases}$$



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Theorem

The worst case time complexity of BINARYSEARCH *on an input size of* n

$=$

# of bits in the binary representation of n.


TREASURE
HUNT
on foot


Hints
&
Tips


Help
Wanted

Analysis of Mergesort in CLRS (# of Comparisions; $a_i : \infty$ not Counted)

(a) Analyze the worst case $W(n)$ and the best case $B(n)$ time complexity of mergesort *as accurately as possible*.
Explore the relation between them and the binary representations of numbers.
Plot $W(n)$ and $B(n)$ and explain what you observe.

(b) Analyze the average case $A(n)$ time complexity of mergesort.
Plot $A(n)$ and explain what you observe.

(c) Prove that: The minimum number of comparisons needed to merge two sorted arrays of equal size m is $2m - 1$.

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$W(n)$: Consider $W(n+1)$

$$W(n) = W(\lfloor \frac{n}{2} \rfloor) + W(\lceil \frac{n}{2} \rceil) + (n - 1)$$

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Theorem

The worst case time complexity of $\textsc{Mergesort}$ *on an input size of* n

$=$

The total $\#$ *of bits in the binary representations of all the numbers* $< n$.

$S(n)$: $\#$ of bits in the binary representations of all the numbers $< n$.

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1		1		1		1
10		10		10		10
11		11		11		11
100		100		100		100
101		101		101		101
110		110		110		110
111		111		111		111
1000	=	1000	+	1000	+	1000
1001		1001		1001		1001
1010		1010		1010		1010
1011		1011		1011		1011
1100		1100		1100		1100
1101		1101		1101		1101
1110		1110		1110		1110

Problem (Area-Efficient VLSI Layout)

Embed a complete binary tree of n nodes into a grid with minimum area.

- Complete binary tree circuit of

$$\#\text{layer} = 3, 5, 7, \ldots$$

- Vertex on grid; no crossing edges
- Area:

$$\underbrace{A(n)}_{\text{area}} = \underbrace{H(n)}_{\text{height}} \times \underbrace{W(n)}_{\text{width}}$$

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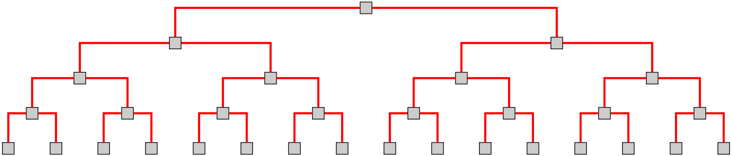
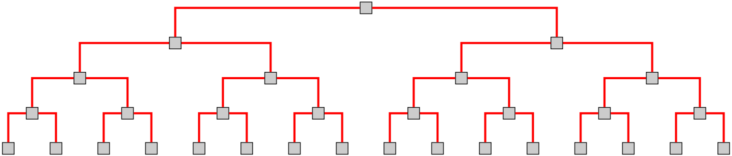
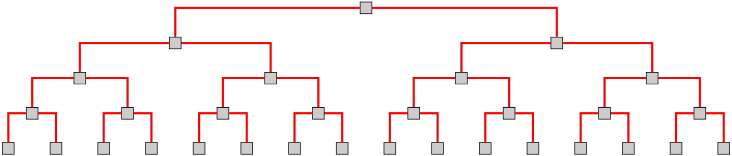




$$H(n) = H(\frac{n}{2}) + \Theta(1) = \Theta(\log n)$$



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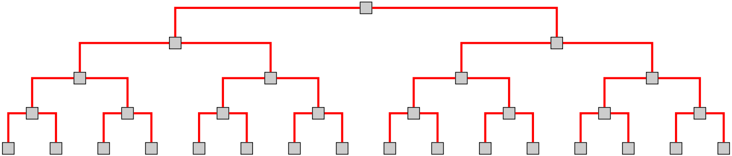

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$$\boxed{A(n) = \Theta(n \log n)}$$

$$Q : \boxed{H(n)} \times \boxed{W(n)} = n$$

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$$\frac{n}{\log n} \times \log n$$

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$$H(n) = \Box H(\frac{n}{\Box}) + O(\Box)$$

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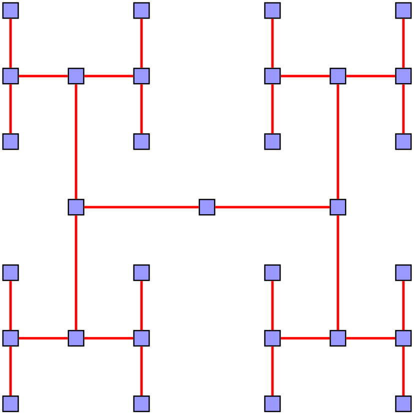
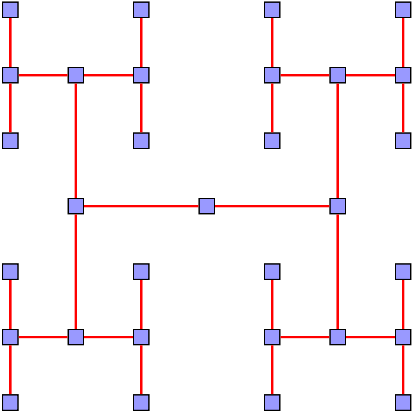
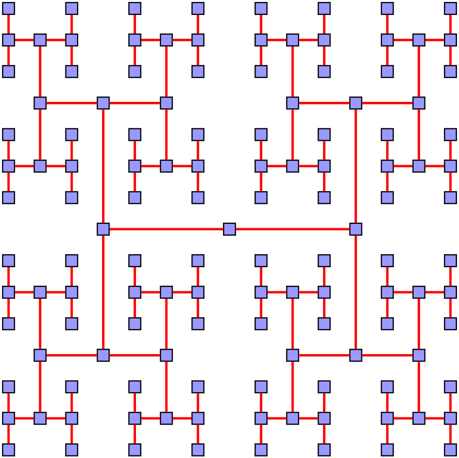
$$\boxed{H(n) = 2H(\frac{n}{4}) + \Theta(1)}$$



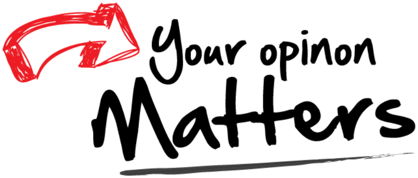


H-layout



"VLSI Theory and Parallel Supercomputing", Charles E. Leiserson, 1989.


Thank
You!



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