计算机问题求解 – 论题1–11 - 有限与无限

2013年12月24日

“聪明的经理”、“非常聪明的经理”和“非常非常聪明的经理”

问题1：

你能给我们讲讲这个故事吗？

问题2：

你能够用集合与函数的概念来解释“小朋友数糖果”的过程吗？

集合的等势

To make precise what it means for two sets (even two infinite sets) to have the same number of elements, we need a definition. We say that a set A is **equivalent** to a set B if there exists a bijection $f : A \to B$. We write $A \approx B$ for A is equivalent to B. (Other authors use the words equipotent or equinumerous.)

问题3：

你原来脑海中的"两个集合元素一样多"的概念是什么样的呢？对无穷集合适用吗？

有限集

. We say that a set S is finite if either $S = \emptyset$ or if S is equivalent to the set $\{1, 2, 3, \ldots, n\}$ for some positive integer n.

更加数学化的表述：(每一个自然数也是一个集合)

空集记为0；

如果k是自然数，则其“后继”为: $k \cup \{k\}$ 。

于是：

有限集就是与某个自然数等势的集合。

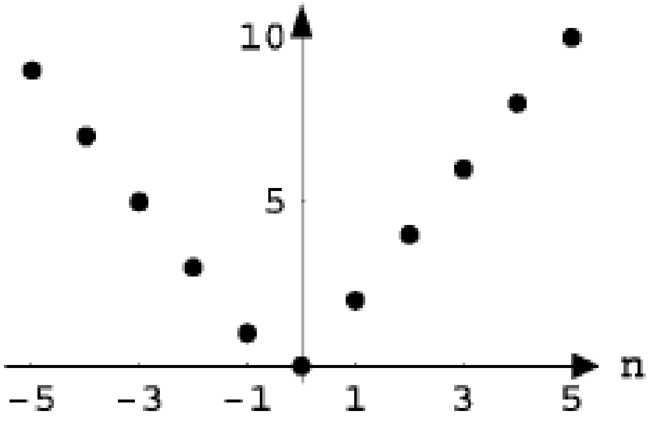
问题4：

什么是无限集合？

自然数集与整数集等势

$f : \mathbb{Z} \to \mathbb{N}$ explicitly as follows:

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -(1+2x) & \text{otherwise} \end{cases}$$



"排队"：

0, -1, 1, -2, 2, -3, 3, -4, 4, -5, 5, …

问题5:

“…-3,-2,-1,0,1,2,3,…”不能算“排好队”了，为什么？

关于双射的证明 (1)

$f : \mathbb{Z} \to \mathbb{N}$ explicitly as follows:

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -(1+2x) & \text{otherwise} \end{cases}$$

注意：
不能遗漏了case 3!

Proof that f is one-to-one.

Let $m, n \in \mathbb{Z}$ and suppose that $f(m) = f(n)$.

Case 1. Suppose that $m \geq 0$ and $n \geq 0$. Then $f(m) = 2m$ and $f(n) = 2n$. Thus $2m = 2n$, and therefore $m = n$.

Case 2. Suppose that $m < 0$ and $n < 0$. Then $f(m) = -2m - 1$ and $f(n) = -2n - 1$. Thus, $-2m - 1 = -2n - 1$, and therefore $m = n$.

Case 3. Suppose that one of the two, say m, is nonnegative, and the other is negative. Then $f(m) = 2m$ and $f(n) = -2n - 1$. Thus $2m = -2n - 1$. But this means that an even number, $2m$, is equal to an odd number, $-2n-1$, which is impossible.

Therefore, if $f(m) = f(n)$, only case 1 and case 2 can occur. In either of these cases, we have shown that $m = n$. Thus f is one-to-one. ■

关于双射的证明 (2)

$f : \mathbb{Z} \to \mathbb{N}$ explicitly as follows:

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -(1 + 2x) & \text{otherwise} \end{cases}$$

Proof that f maps $\mathbb{Z}$ onto $\mathbb{N}$.

Let $k \in \mathbb{N}$. If k is even, then $k = 2m$ for some $m \in \mathbb{Z}$ with $m \geq 0$. Thus, $m \in \mathbb{Z}$ and $f(m) = 2m = k$. If k is odd, then $k + 1$ is even. Hence $m = (k + 1)/(-2) \in \mathbb{Z}$. Since $k \geq 1$, we have $m < 0$. Thus, $f(m) = -2m - 1 = -2((k + 1)/(-2)) - 1 = k$. We conclude that for all $k \in \mathbb{N}$, there exists $m \in \mathbb{Z}$ such that $f(m) = k$. Since $f : \mathbb{Z} \to \mathbb{N}$ is a well-defined function, f maps $\mathbb{Z}$ onto $\mathbb{N}$. ■

无穷不仅仅是“很多很多”

伽利略悖论：

整体与局部“一样大”！

Theorem 20.6.
Let A, B, C, and D be nonempty sets. Suppose that $A \cap B = \emptyset$, $C \cap D = \emptyset$, $A \approx C$, *and* $B \approx D$. *Then* $A \cup B \approx C \cup D$.

Corollary 20.8.
Let A and B be disjoint sets. If A and B are finite, then $A \cup B$ *is finite.*

Theorem 20.10.
Let n be a positive integer. Then every subset of $\{1, 2, 3, \ldots, n\}$ *is finite.*

Corollary 20.11.
Let S be a finite set. Then every subset of S is finite.

Theorem 20.12.
The union of two finite sets is finite.

顺便问一句：$\{1,2,3,\ldots,n\}$ 本身是有限集为什么不要证明？

问题6:

你能在这里找到几何证明中“加辅助线”的感觉吗？

问题7:

为什么"鸽巢原理"在证明一个集合是无限集合时有关键的应用？

鸽巢：“证明策略”和“数学定理”

In its popular form, the principle says that *if there are more pigeons than holes, then at least one hole is the home of more than one pigeon.*

Theorem 21.2 (Pigeonhole principle).
Let m and n be positive integers with $m > n$, and let f be a map satisfying $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$. Then f is not one-to-one.

The proof of the pigeonhole principle summarizes much of what you learned: mathematical induction, proof in cases, and one-to-one functions.

问题8：

你能简述一下这个证明的基本思路吗？

问题9：

对于“一对一”性质的满足，一个函数与其在定义域的某个真子集上的“限制”相互是什么关系？

自然数集是无限集

反证法

Suppose to the contrary that $\mathbb{N}$ is finite. Since $\mathbb{N} \neq \emptyset$ there exists an integer m and a one-to-one mapping, g, of $\mathbb{N}$ onto $\{1, 2, \ldots, m\}$. Now $\{1, 2, \ldots, m+1\} \subseteq \mathbb{N}$, so we may consider the restriction $g|_{\{1,2,\ldots,m+1\}} : \{1, 2, \ldots, m+1\} \to \{1, 2, \ldots, m\}$. The pigeonhole principle (Theorem 21.2) implies that $g|_{\{1,2,\ldots,m+1\}}$ is not one-to-one. This, in turn, implies (as you surely showed in Exercise 20.9) that g is not one-to-one, contradicting our choice of g. Therefore, it must be the case that $\mathbb{N}$ is infinite. ∎

有限集合的“势”(cardinality)

Theorem 21.6.

Let A be a nonempty finite set. There is a unique positive integer n such that $A \approx \{1, \ldots, n\}$.

正因为这样的*n*的唯一性，可以定义集合的“势”。

问题10：

如果不唯一，怎样能构造出与“鸽巢原理”的矛盾来？

可数集

An infinite set A is said to be **countably infinite** if $A \approx \mathbb{N}$.

A set is **countable** if it is either finite or countably infinite.

a nonempty set A is countable if and only if there exists a one-to-one function $f : A \to \mathbb{N}$.

注意：

这个性质使得我们可以不区分有限或无限可数；

通常找一个一对一的函数比找一个双射容易。

可数集的子集是可数集

可以比拟为“辅助线”的定理：

Every subset of $\mathbb{N}$ *is countable.*

“*N*与其任一无限子集*T*之间存在双射”证明的基本思路：

1. 建立一个函数$f : f(k)=T$ 中第k个“最小”元素；
2. 证明f 是一对一的：函数值构成严格递增序列；
3. 证明f 是满射：对T 中任何元素s , 假设比s 小的元素有n 个，则$f(n+1)=s$。

有理数集是可数集

我们似乎没法“数”有理数，但是有理数集是“可数”集。

I see it but I do not believe it.—Georg Cantor

We will begin by showing that $\mathbb{Q}^+$ is countable. We define $f : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$ as follows. Write each member of $\mathbb{Q}^+$ as p/q where $p, q > 0$ and p/q is in reduced form; that is, p and q have no positive common factor other than 1. Now define $f(p/q) = (p, q)$. Because p/q is in reduced form, f is well-defined and one-to-one. Since $\mathbb{N} \times \mathbb{N}$ is countable (Theorem 22.8), and $f(\mathbb{Q}^+)$ is a subset of it, we know from Corollary 22.4 that $f(\mathbb{Q}^+)$ is countable. Hence $\mathbb{Q}^+$ is countable. Now the set of negative rationals, $\mathbb{Q}^-$, is equivalent to $\mathbb{Q}^+$. Since $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$, and we have a finite union of countable sets, we use Corollary 22.7 to conclude that $\mathbb{Q}$ is countable. Since $\mathbb{Q}$ is infinite we know that it is countably infinite. ■

但是实数集**不是**可数集！

Cantor's diagonalization argument

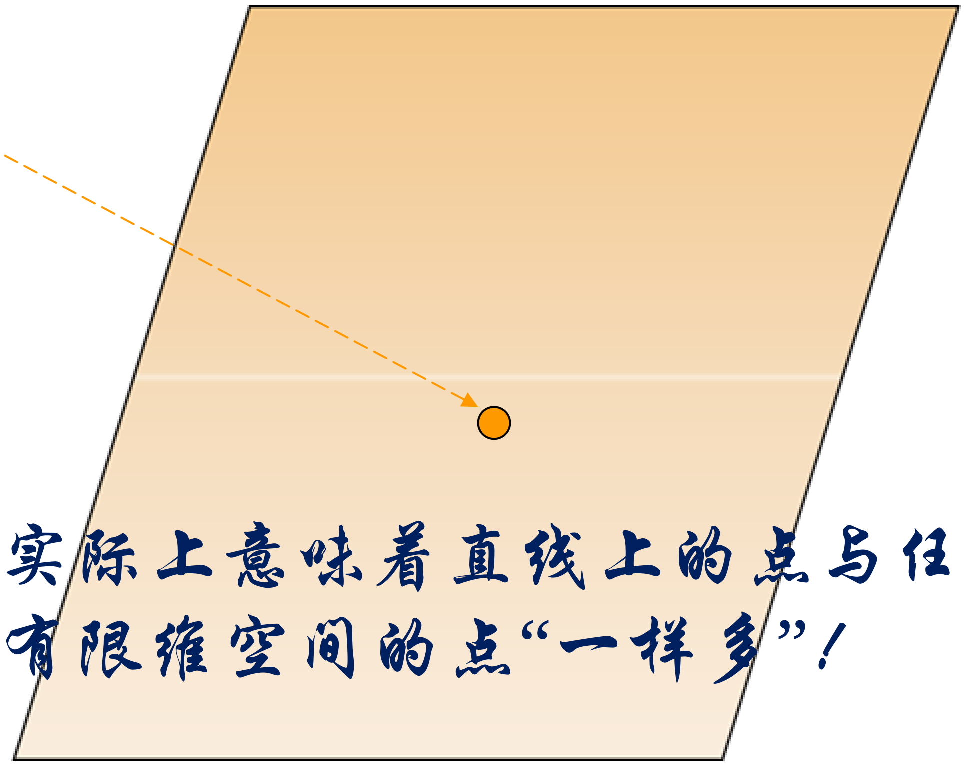
$$
\begin{aligned}
f(1) &= 0.\mathbf{a_{11}}a_{12}a_{13}\ldots \\
f(2) &= 0.a_{21}\mathbf{a_{22}}a_{23}\ldots \\
f(3) &= 0.a_{31}a_{32}\mathbf{a_{33}}\ldots \\
&\vdots \\
f(n) &= 0.a_{n1}a_{n2}a_{n3}\ldots\mathbf{a_{nn}}\ldots
\end{aligned}
$$

矛盾 ⟶ $b = 0.\mathbf{b_1 b_2 b_3}\ldots$

直线上的点集与平面上的点集等势

$$\begin{cases} 0.a_1a_2a_3\ldots\ldots \\ 0.b_1b_2b_3\ldots\ldots \end{cases}$$

$0.a_1b_1a_2b_2a_3b_3\ldots\ldots$



这实际上意味着直线上的点与任意有限维空间的点"一样多"！

康托尔定理 – 没有“最大”的集合

- 任何集合与其幂集不等势　　即：$A \not\approx \rho(A)$
  - 证明要点：

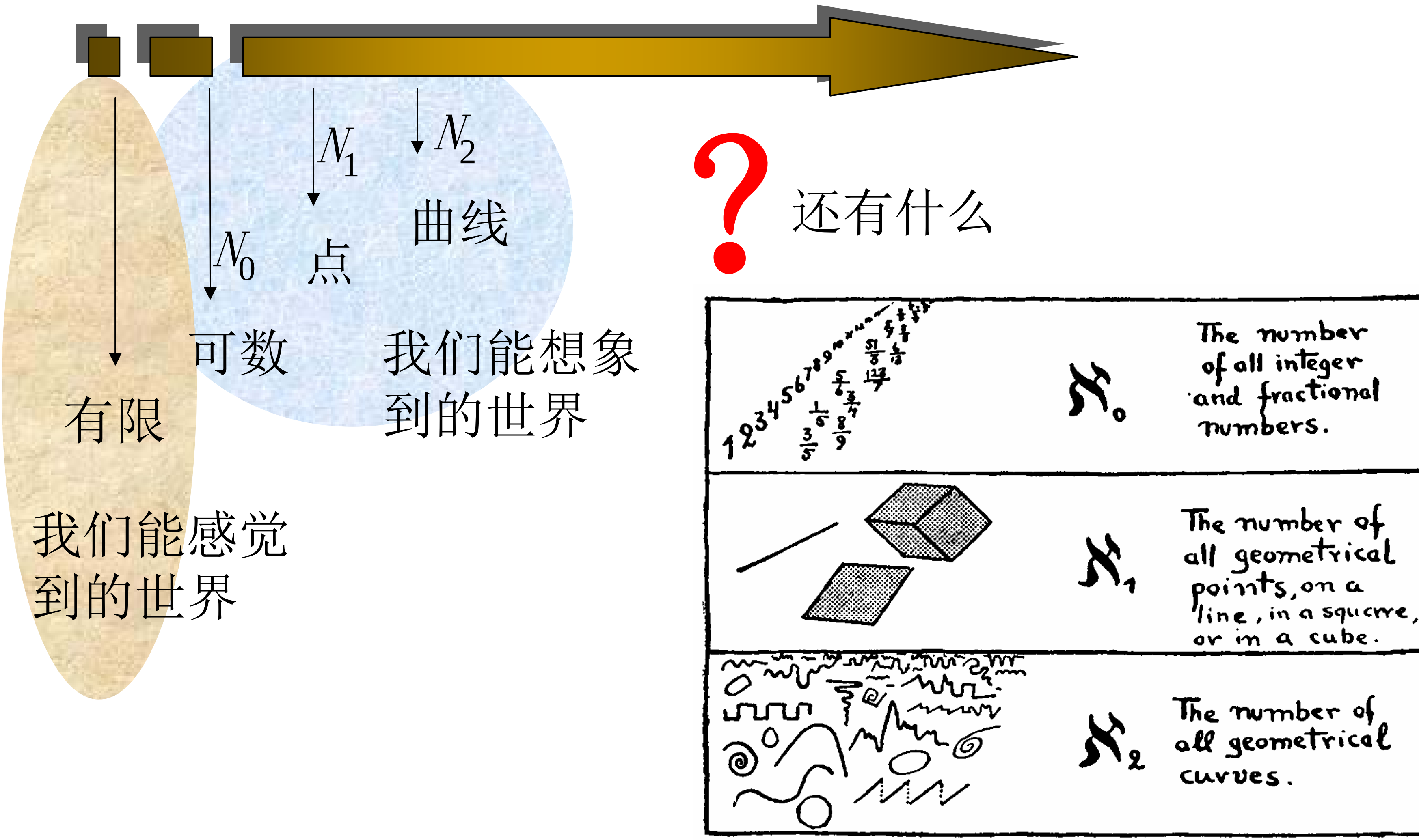
设g是从A到ρ(A)的函数，构造集合B如下：

$$B=\{x \mid x\in A, 但 x\notin g(x)\}$$

则$B\in\rho(A)$，但不可能存在$x\in A$，能满足$g(x)=B$，因为，如果有这样的x, 则$x\in B$ iff. $x\notin B$。

因此，g不可能是满射。

集合的“大小” - 基数



有限
我们能感觉到的世界
N_0
可数
N_1
点
N_2
曲线
我们能想象到的世界
?
还有什么
$\aleph_0$
The number of all integer and fractional numbers.
$\aleph_1$
The number of all geometrical points, on a line, in a square, or in a cube.
$\aleph_2$
The number of all geometrical curves.

家庭作业

- UD
  - problems: 20.4, 20.8-10;
  - problems: 21.7, 21.9-11, 21.16-19;
  - problems: 22.1-3, 22.6, 22.9
  - problems: 23.2-3, 23.10